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G
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Obsah – Contents

SLOVENSKÁ SPOLOČNOSŤ

SG

PRE GEOMETRIU A GRAFIKU

Nezisková vedecká spoločnosť pre rozvoj geometrie a počítačovej grafiky

zaregistrovaná dňa 13.5.2002 na Ministerstve vnútra SR ponúka
všetkým záujemcom individuálne alebo kolektívne členstvo.
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d) podieľať sa na organizácii vedeckých podujatí, konferencií, seminárov a sympózií o geometrii a počítačovej grafike

e) publikovať vedecký časopis s názvom G venovaný geometrii a grafike

f) rozvíjať vlastnú edičnú a publikačnú činnosť

g) získať priazeň a členstvo organizácií aj jednotlivcov.

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Golden section: applications in domain of landscape architecture

Biljana Jović, Daniela Velichová, Milena Cvjetić

Abstrakt

Článok pojednáva o vzťahoch medzi prírodnými štruktúrami a zlatým rezom a analyzuje niekoľko príkladov aplikácií zlatého rezu v oblasti krajinnej architektúry. Prezentované sú výsledky výskumu týkajúce sa podstaty geometrickej konštrukcie zlatého rezu a predstavené sú niektoré ukážky aplikácií konštrukcie zlatého rezu pomocou prvkov vizuálnej estetiky v krajinnej architektúre, ktoré sú základnými prvkami kompozície. Cieľom tohto článku je systematizácia spomenutých prvkov z pohľadu využitia zlatého rezu a ich aplikácia na konkrétnych príkladoch z oblasti krajinnej architektúry.

Kľúčové slová: zlatý rez, geometria, bionika, príroda, krajinná architektúra

Abstract

This paper deals with the analysis of the relationship between natural structures and golden cross sections, and application of the golden cross section in the domain of landscape architecture. The aspects and research results shown in this paper are concerning the geometric construction of the golden section and its applications by the elements of visual aesthetics in landscape architecture as the basic elements of the composition. The aim of the paper is the systematization of elements from the aspect of using the golden section, as well as the application on concrete examples in domain of landscape architecture.

Key words: golden section, geometry, bionics, nature, landscape architecture

1 The concept of a golden section and the number φ

In the art of proportions, relations of size (lines, surfaces, shapes) are in one form, i.e. compositions. Proportions develop and affirm the unity of the whole. A good proportion means that each element of the composition is in a harmonic relationship.

The basic task of the theory of proportions is to create visual work and balance. In order to satisfy our needs, humans have been producing, from ancient times, products and objects that, apart from function and purpose, must be in a certain dimension, above all in relation to man as their beneficiary.

The only natural arithmetic proportion that could be obtained with just two elements is expressed by the following formula

$$(a + b) : a = a : b$$

In this dimension, the smaller size refers to the larger as the larger to the whole. Thus, the golden cross is the ratio of the quantities where the smaller part refers to the larger, as larger to the whole, or vice versa where the greater part refers to the smaller as well as the whole to the greater part.

The term "golden section" seems to be used for the first time in the year 1835 by German mathematician Martin Ohm (in German as *goldener Schnitt* or *der golden Schnitt*), in the second edition of his textbook *Die Reine Elementar-Mathematik* , as quoted in [1] , p. 6. The first known use of this term in English can be found in the paper on aesthetics from the year 1875, written by James Sulleyand published in the 9th edition of the *Encyclopedia Britannica.* The symbol ϕ ("phi"), or sometimes φ, was used by Mark Barr at the beginning of the 20th century in commemoration of the Greek sculptor Phidias (about 490 - 430 BC), as a number of art historians claim that he made extensive use of the golden ratio in his masterpieces.

Golden ratio has connections with various concepts in number arithmetic, such as the Euclidean algorithm for computing the greatest common divisor of two integers, or concept of continued fractions, and others. It is an irrational number, which can be calculated sequentially, as the ratio of two consecutive numbers of Fibonacci sequence: the bigger the pair of Fibonacci numbers has been chosen, the closer the approximation of golden ratio could be achieved. Fibonacci sequence is defined by an easy rule - the next number is found by adding up the two numbers before it, so starting with numbers 1 and 1, the few first Fibonacci numbers are $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots 144, 233, 377, \dots\}$ and the sequence of the ratios of two consecutive numbers

$$\{1, 2, 1.5, 1.666, 1.6, 1.625, 1.6153, 1.619, 1.6176, \dots, 1.61805, 1.61802, \dots\}$$

is converging to the exact value of the golden mean.

One particularly interesting property of the golden ratio is that it can be defined in terms of itself, which means

$$\varphi = 1 + \frac{1}{\varphi}$$

and can be rewritten as quadratic equation

$$\varphi^2 - \varphi - 1 = 0$$

with 2 real roots

$$\varphi_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \varphi_2 = \frac{1-\sqrt{5}}{2}.$$

The positive solutions (ratio of two positive quantities) represents the simple formula to calculate the golden ratio value

$$\varphi = \frac{1+\sqrt{5}}{2} = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

and reveals the clue how to extend the square to be a rectangle with the golden ratio. Here is one way how to draw such a rectangle (and line segment of the golden section length), see in Fig. 1:

- Draw a square of size 1 of its sides.
- Find centre *S* of one of its sides.
- Draw a line segment from point *S* to the opposite vertex *C* of the square (it is $\sqrt{5}/2$ in length).
- Revolve line segment *SC* about point *S* so that it coincides with the side *AB* of the square.

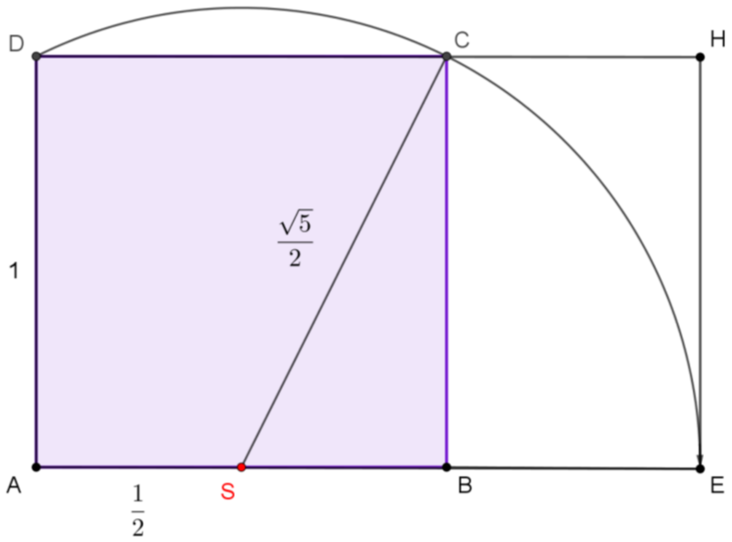


Fig. 1. Construction of golden rectangle from a unit square

Thus irrational number φ can be approximately estimated as

$$\varphi = 1.61803 = 1 + 0.61803 = 1 + \frac{1}{1.61803} = 1 + \frac{1}{\varphi}$$

The above formula can be expanded to the continued fraction

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\varphi}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} .$$

Pentagon, and pentagram - famous as a magical or holy symbol, are hiding the golden ratio. The ratio of the diagonal to the side of a regular pentagon is the Divine Proportion. Moreover, the diagonals create an isosceles triangle (where two of the three sides are equal) with angles of 72 degrees and 36 degrees. This triangle can be reproduced inside itself to infinity (in a "self-developing" manner), as shown in Fig. 2:

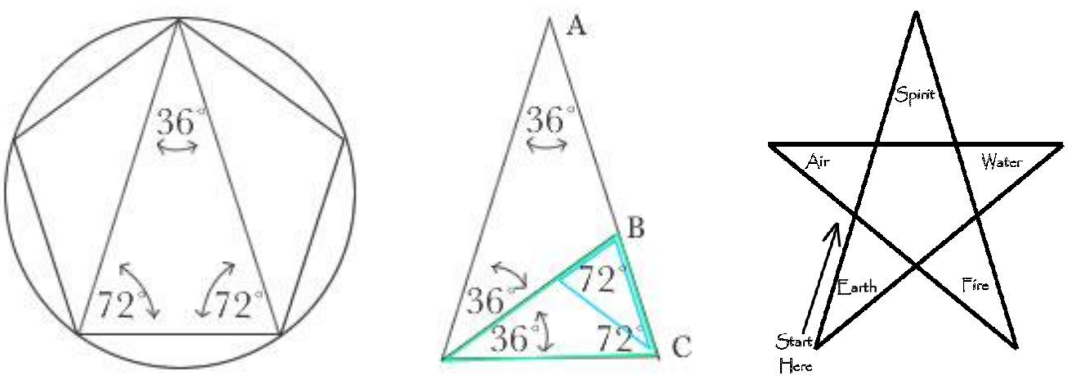


Fig. 2. Regular pentagon, and pentagram

The construction of line, triangle, rectangle, pentagram and, at the end, logarithmic spiral (golden spiral) in accordance with the golden ratio rules, forms the most harmonic shapes found in nature (Fig. 3).

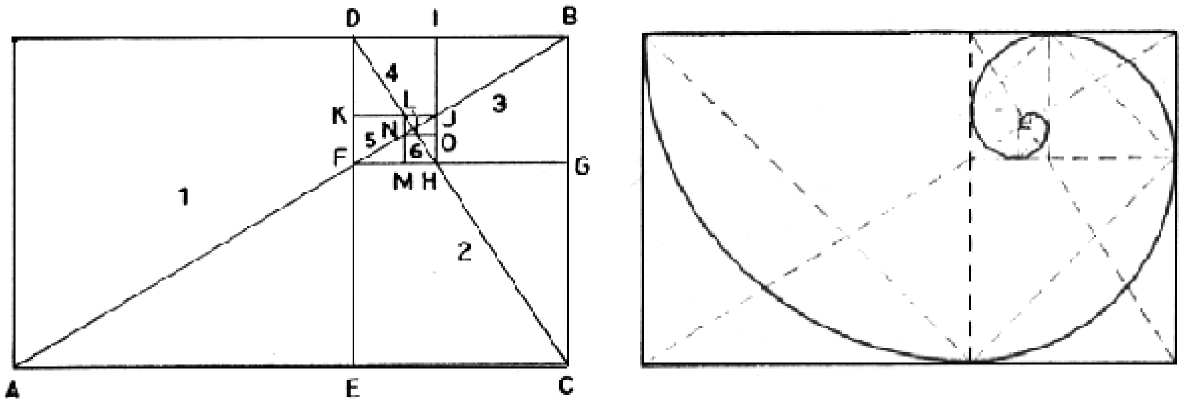


Fig. 3. Construction of the logarithmic spiral in the golden rectangle [http://mathworld.wolfram.com/GoldenRectangle.html]

1.1 Golden ratio in nature and art

1.1.1 Golden ratio in nature

The structure of DNA molecule is in the ratio of the golden cross section. The pulse of a man's heart, shown on the ECG footage, is also in the same relation.

In nature, the phenomenon of distribution of leafs on the plant could be seen governed by the rule of Fibonacci series called phyllotaxis (from Ancient Greek *phýllon* – leaf and *taxis* - position). It occurs in plant species such as american linden, beech, hazelnut, oak, apricot, poplar, pear, willow, almond, etc.

Phyllotaxis is the simplest way of distributing leafs evenly on the plant, where each leaf surface receives a sufficient amount of light from the least occupied space. Also, the arrangement of branches in many species takes place according to Fibonacci’s series of numbers [8].

In addition to this, the germ’s - germination process, from the soil, has the shape of the golden spiral. The blooming sunflower and many other plants from the *Asteraceae* family contain a golden spiral. In addition to them, they could be seen on some of the pines, fruits of broccoli, pineapples, etc. A large number of flowers have a number of petals that correspond to Fibonacci’s numbers as iris or lily, and most flowers have 5 petals (violet, tulips, etc.). The rings of growth on some palm trees form a golden spiral.

Beside in flora, the golden cross section is very present in the animal kingdom also. The ratio of females and males in the bee hive is equal to the number φ. Many shells of snails, especially nautilus, have the shape of a golden spiral (Fig. 2). Relations of body parts of many animals are found in the golden cross section [8].

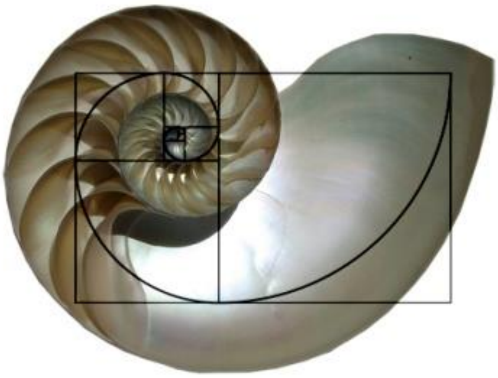
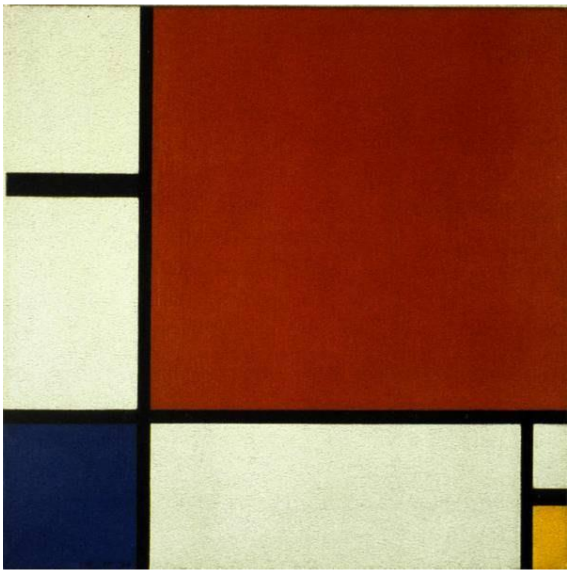



Fig. 4. Golden section in the nautilus shell and example of flower with 5 petals [https://www.flowersforums.com/please-identify-this-pink-5-petal-flower-found-growing-near-water-in-southw...-34786.html http://www.photoshoptutorialsandtips.com/wp-content/uploads/2010/10/nautilus_withoverlay.jpg]

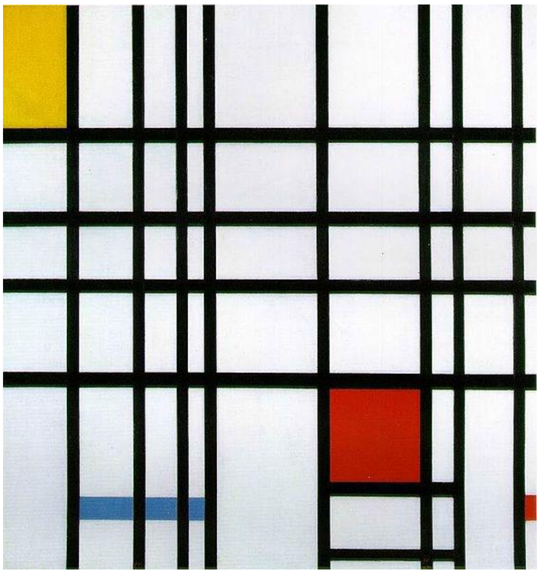
Concerning human proportions, the most well-known proportional marking of a human body can be found in Leonardo da Vinci's study called "Homo Vitruvius" (The Vitruvius Man). In the 20th century, French architect Le Corbusier, in 1945, introduced the theory and practice of the golden-cross-section proportions applied to man as well as the modular system - Modulor.

1.1.2 Golden ratio in art

The golden section was used in the art since the age of Ancient Egypt, and it could be further found in every sphere of art and every art epoch.



a)



b)

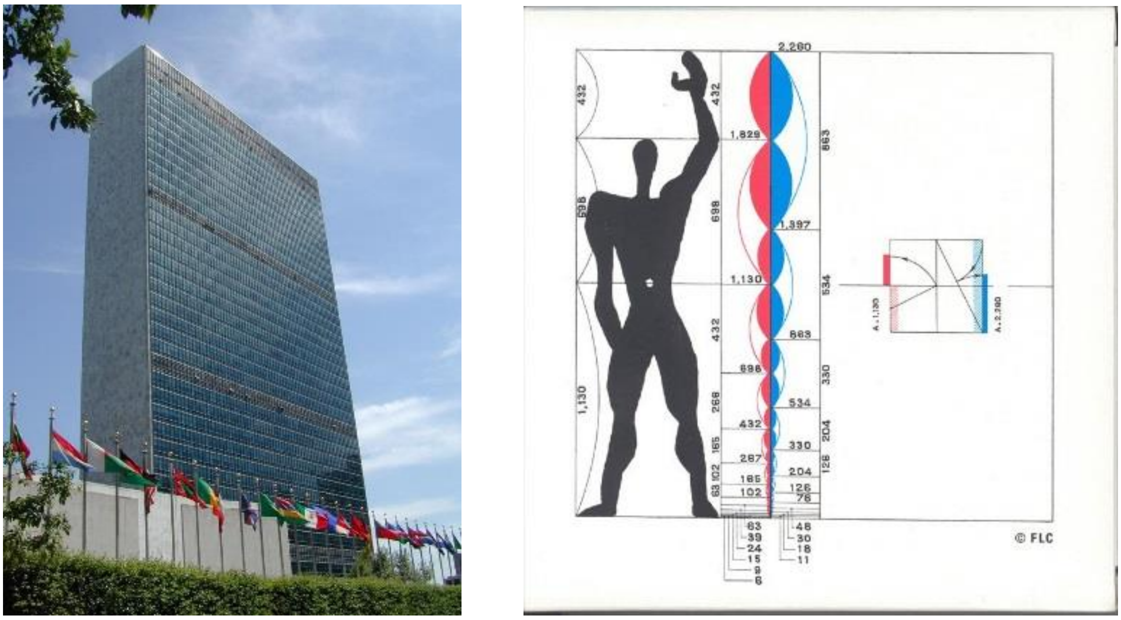
Fig. 5. a) "Composition II RBY", Mondrian, b)"Composition II RBY", Mondrian [http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html]

In addition to Leonardo da Vinci, the artist whose works are a clear association with the golden section is definitely Pit Mondrian, who developed the direction in abstract art – Neoplasticism [2]. The base of the image is white, and there are black straight lines over it, sometimes filled with only three basic colours: red, blue and yellow. By using basic colours and simplest shapes, Mondrian emphasizes that any shape could be created using basic geometric shapes and that any colour can be obtained by using combination of these three basic colours Mondrian claimed that painting and geometry are inextricably linked as well as the nature and Neoplasticism, see Fig. 5. One of the most commonly used forms in his paintings is a gold rectangle [2].

1.1.3 Golden ratio in architecture

The application of the golden section in architecture is reflected in works that remained as a cultural and historical heritage, starting from the Pyramid in Giza, to modern architectural objects. The theory of architecture explains the beginnings, but also the development of the theory of golden cross-section through the anthropometric understanding of architecture [4].

In the 19th century, Ernest Neufert deals in detail with the golden section as an architectural proportional system, referring to historical heritage. Together with Cajsing, they are considered to be the greatest advocates of anthropometric architecture [5]. In the 20th century, the French architect Le Corbusier did not see the application of the golden section in architecture as "natural rhythm" in proportion, but in the design organization, the façade, in particular, see in Fig. 6 b), [5].



a) b)

Fig. 6. a) Facade of the United Nations Building in New York, divided into three gold rectangles, Le Corbusier [https://es.wikiarquitectura.com/sede_de_la_onu_ny_4/]
b) "Modulor", Le Corbusier [http://www.fondationlecorbusier.fr/corbuweb/morpheus.]

The most important Le Courbusier's work is Modulor, a system of proportions of a man in a golden cross section, representing a modern interpretation of "Homo Vitruvius", including certain examples of proportions in architecture, Fig. 6b). The proportions of the golden section and the continuous division of Le Courbusier emphasizes using three basic colors, similar to Mondrian [5].

2 Bionics

Bionics is a young scientific discipline, which got its name from the Greek word *βίος*, meanings element of life. The essence of the bionic approach is the study of biological methods and systems that are in nature in order to apply this knowledge in design, or in the tasks of modern engineering technology [6].

What Nature has created represents most commonly the most practical, the most ideal and the most economical solutions, which gave the engineers the idea to use them in solving increasingly complex problems [3].

The practical realization of the bionic approach takes place on three levels, i.e. there are three methodologies within the bionics: biological, mathematical and technical.

Biological bionics uses knowledge most commonly in medicine, botany and zoology to extract the principles of the functioning of the observed organisms, which can establish a link to the technical problem that engineers find. Then the mathematical approach to processing this biological content, so-called biological modelling, where mathematical models represent a faithful copy of the biological content or process that is vital to the problem posed. Finally, technical bionics aims to provide technical realization and practical application of the mathematical model [7].

One of the first serious papers on the bionic approach is the book On Growth Of Form, revised in 1917 by English biologist Darsy Wentworth Thompson. In this book, Thomson expresses the bold claim that the organic world is equally "mathematically" as well as non-organic. Thomson saw the hidden mathematical basis and assumed that the geometric, or mathematical principles on which the organic world functions, really exist. Namely, in his book, he has devoted the entire chapter to numerous examples of geometric laws that appear in the plant world. By studying the botanical world, he noticed the appearance of Fibonacci numbers in the schedule, as well as in number, round and cup leaves, leaves, branches. He also studied the appearance of a logarithmic spiral and a proportional division in the animal world. On the basis of numerology and geometry, which is obviously present in the living world, Thompson made conclusions and rejected the possibility of a random presence of mathematical relations and principles in the living world [8].

Nowadays, biomimetics could be the most appropriate term used in landscape architecture as well as domain in architecture and civil engineering. Biomimetics is a rich design tool that interprets the natural processes and forms transferring them to the artificial creations according to Gruber [9]. Biomimetic approach does not separate fixed entities like form, function, structure and material, but unites them, and is defined as a semi-organic composition. By copying natural models, applying geometric principles and biological knowledge, it is possible to produce spatial structures that are stratified, variable, and connected [10].

2.1 Examples of the golden section in the landscape architecture

The most famous examples of using golden section in domain of landscape architecture follows. **Ma Wan Park** is a park on Ma Wan Island (Ma Wan, New Territories, Hong Kong), the main entrance to Ma Wan Park is the Golden Mean Plaza. The architect of this square intended to express harmony between nature and human creation. Fig. 7a), [11].

Eastwoodhill arboretum (Gisborne, New Zealand) is the national arboretum of New Zealand. The Fibonacci spiral, which is located as one spatial entity in this arboretum, is a memorial to H. B Williams, a man in charge of the existence of this arboretum, Fig.7b), [12].

a) b)

Fig. 7. a) Nautilus shell sculpture – Ma Wan park
[https://upload./wikimedia/commons/c/c9/Ma_Wan_Park_Gold_Mean_Plaza.jpg]
b) Fibonacci spiral in garden
[http://www.eastwoodhill.org.nz/assets/Education/Fib-Spiral.jpg]

a) b)

Fig. 8 a) The California Polytechnic State University-Engineering Plaza
[http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibInArt.html#parthenon]
b) Using third-party Golden section rules in the parterre, Villandry Castle, France [http://www.fluidr.com/photos/guilminou/7297407762]

California Polytechnic State University - Engeneering Plaza -the geometric shapes used are gold spiral and gold triangle and gold rectangle, Fig. 8a). Apart from geometric, this is also an excellent example of the use of the golden section in a symbolic meaning, bearing in mind the scientific character of the faculty, [13].

The use of the gold cross-section in the formation of a parterre could be found in the Renaissance, and especially in the French Baroque Gardens, see Fig. 8b).

Fig. 9. Orpheus project – Pyramid and inverse pyramid relationship [https://www.kimwilkie.com/uk/orpheus-at-boughton/]

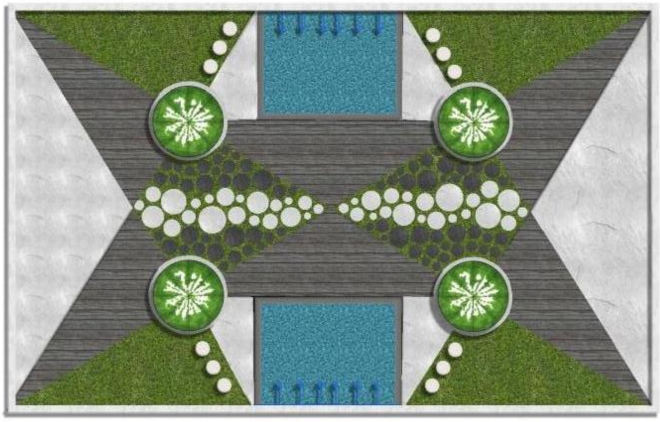
Boughton house and Orpheus project – Boughton is an English formal garden, formed between 1685 and 1725. It is one of the rare examples of geometric garden in Britain, Fig. 9.

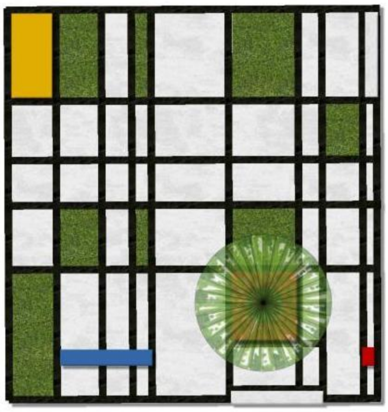
The Orpheus project is a modern addition to the garden, designed by architect Kim Wilkie. Analysing the geometry of this garden, author of the project noticed the use of proportions of the golden section in the composition, and decided to use it in the design of new contents [14].

2.2 Visual – aesthetic elements in landscape design

The composition of certain space could be well perceived by understanding the elements and principles of visual aesthetics. As the process for landscape design in this paper research is done from geometric point of view. The basic visual-aesthetic elements that are analysed are: point, line, shape and size [15].

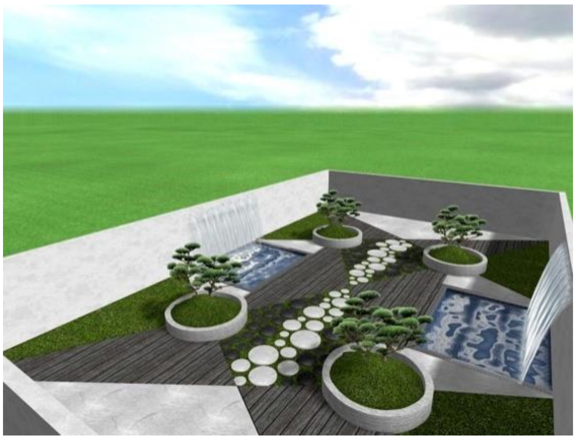
The examples – design concepts of atrium gardens of arbitrary dimensions, foreseen in the ambiance of the business-commercial complex, are presented in two variants with the use of golden section (Fig. 10 and Fig. 11).

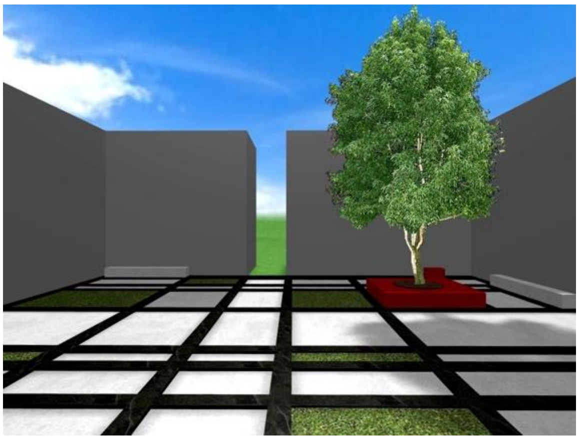




a) b)

Fig. 10. Top view a) Atrium no. 1 – symmetry, b) Atrium no. 2 – asymmetry design concept



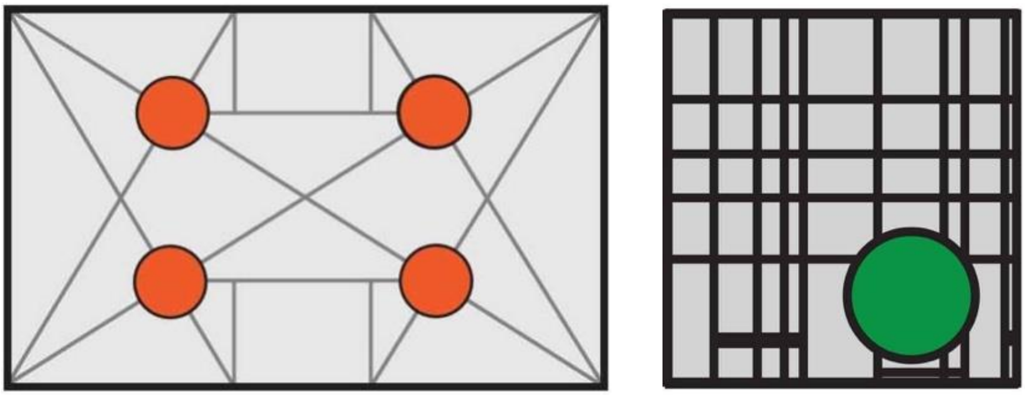


a) b)

Fig. 11. 3D model a) Atrium no. 1 – symmetry, b) Atrium no. 2 – asymmetry

2.2.1 Point

The application of the golden section in landscape architecture can be achieved through focal points, whether they are used as a measure of symmetry or asymmetry in the space.



a) b)

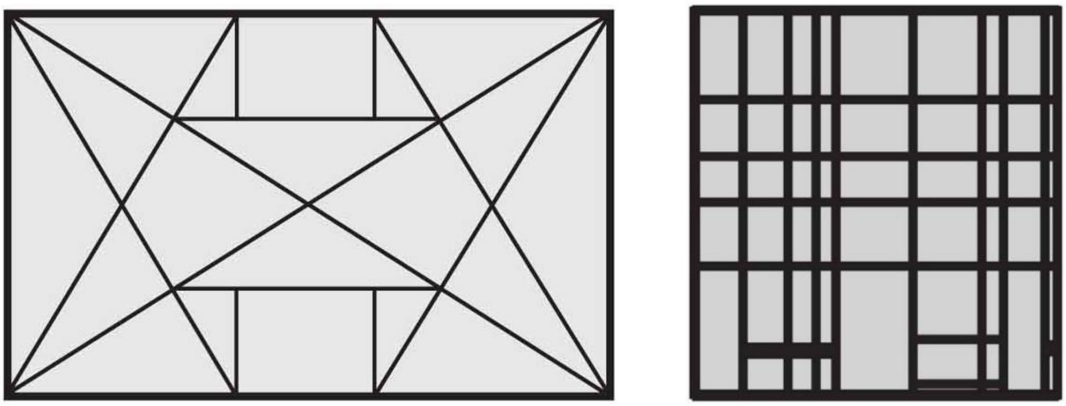
Fig. 12. Focal points a) Atrium no. 1 – symmetry, b) Atrium no. 2 – asymmetry

Atrium number 1 (Fig. 12 a)) is designed according to the pattern of the golden section known as the "eyes of the rectangle". In this case, the focal points are four bonsai trees of black pine, raised in build planters. They form elements of a spatial composition, retaining its symmetry and harmony.

Atrium number 2 (Fig. 12 b)) for example has one of the famous Mondrian's compositions "Red, Yellow, Blue". The center is displaced, but the harmony of the asymmetric composition is achieved using a golden section. The center of the painting composition is represented in the space by a raised planter with a tree. The point becomes the center of visual gravity, both horizontally and vertically.

2.2.2 The line

In landscape architecture, line items could be considered as line elements, such as paths, alley, live fences, borders, walls, channels, paving slabs, etc., as long as their role implies one direction – one line [16].



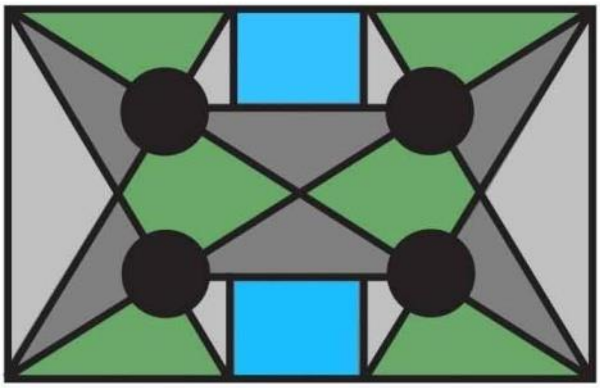
a) b)

Fig. 13. Line composition a) Atrium no. 1 – symmetry and b) Atrium no. 2 – asymmetry

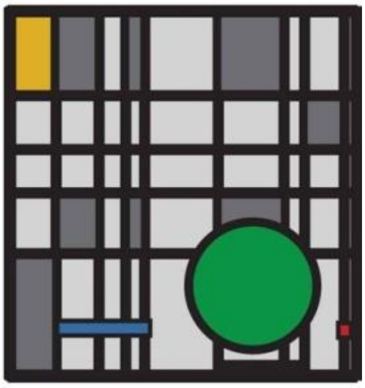
In example of atrium no. 1 (Fig. 13 a)), lines represent the boundaries of different surfaces – stone, wood, water, which are in the same vertical plane. In example of atrium no. 2 (Fig. 13 b)), lines represent change in paving materials, making the geometric network the basis of the composition, defining the space and directions of movement.

2.2.3 Shape (surface)

The shape is one of the basic elements of the architectural composition. The three basic types of shapes are geometric, natural, and abstract. Golden rule can be applied in every one of them. Given the spatial perception of shapes in landscape architecture, the golden rule should be applied to the geometric compositions in the horizontal plane that could be seen and understood by users of space [16].



a)



b)

Fig. 14. Geometric shapes: a) Atrium no. 1 – symmetry and b) Atrium no. 2 – asymmetry

As illustrated in the example of atrium (Fig. 14), this is most easily achieved by various shifts in paving materials, colours, or in the composition of the parterre surfaces.

In the case of atrium number (Fig. 14a)), positioning the four points in the space creates the impression of the golden rectangle. Parterre represents a network of triangles, which does not define a contour, but different materials that fills them.

In the case of atrium no. 2 (Fig. 14b)) the paving lines divide the space on a series of golden rectangles, which are clearly visible and recognizable to users.

2.2.4 Size

Finally, the ratio of the sizes between the individual elements is proportion, and the systematic care of relations is one of the principles of visual aesthetics [16].

3 Discussion

In this paper, The Golden section is considered from the standpoint of several disciplines: geometry, art, architecture, biomimetics and landscape architecture. An important feature of the golden section is its ability to be applied both in plane and in space and in symmetry and asymmetry, especially as element of balance into asymmetric compositions. This biomimetric approach to the design process could serve as an example of a new aspect of the use of geometry in landscape architecture.

The paper deals with the geometric analysis of the composition in the plane, but there are numerous examples of the use of golden section in space: with plants, garden and architectural elements. The aspect of using the golden section in landscape architecture requires much more extensive research, which goes beyond the scope of this paper, and leads us to direction of further more detailed researching that starts from this point.

Acknowledgement

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Teaching calculus with Maria Gaetana Agnesi

Paola Magnaghi-Delfino, Tullia Norando

Abstrakt

V roku 2018 uplynulo tristo rokov od narodenia Márie Gaetany Agnesi, matematičky a dobrodinky, ktorá sa narodila v Miláne (v Taliansku). Preskúmali sme *Analytické inštitúcie*, hlavné dielo Márie Gaetany, ktoré venovala vzdelávaniu študentov. Sme presvedčení, že študenti si pred začatím štúdia na vysokej škole môžu osvojiť základné matematické pojmy diferenciálneho počtu pomocou metód a ideí navrhnutých v tejto knihe datujúcej sa do odobia zrodu analýzy. Stačilo by využiť niektoré z množstva návrhov a príkladov obsiahnutých v knihe od Agnesi [1].

Kľúčové slová: dejiny matematiky, cykloida, versiéra, Mária Gaetana Agnesi

Abstract

In 2018, we celebrated the three hundredth anniversary of the birth of Maria Gaetana Agnesi, mathematician and benefactress, born in Milan (Italy). We have examined the *Analytical Institutions*, the main work of Maria Gaetana, that she dedicated to students' education. We think that pre-university students can acquire the fundamental mathematical ideas in Differential Calculus using methods and ideas proposed in the books that go back to the origins of the Analysis. From this point of view, we can use many suggestions and examples, contained in Agnesi's Books [1].

Key words: history of mathematics, cycloid, versiera, Maria Gaetana Agnesi

To the first woman in the Western world to have achieved a reputation in mathematics
Maria Gaetana Agnesi (1718 – 1799)

1 Introduction

The seventeenth century is one of the most exciting periods in the history of mathematics. The first half of the century saw the invention of the analytic geometry and the discovery of new methods for finding tangents, areas and volumes. These results set the stage for the development of the calculus during the second half of the century. The plane curves, both those known from antiquity and those discovered in recent times, played a central role and were used by nearly every mathematician of the time as examples for demonstrating new techniques. We must remember that the concept of function is a discovery of the nineteenth century, whereby the known curves were defined in a geometric or mechanical way. Among these curves, the cycloid had a particular role in the solution of two important problems: to find the fastest path between two positions and to build a more reliable watch.

The cycloid is the curve traced out by a point on the circumference of a circle, called the generating circle, which rolls along a straight line without slipping. Galileo originated the term *cycloid* and was the first to make a serious study of the curve, despite the fact that some mathematical historians argue that the curve was known since antiquity [11].

Scientific discoveries were only known through the reading of the very few scientific journals, all in Latin, or through epistolary contacts. The new science spread in scientific circles and, while arousing much curiosity, it was only heritage of scientists.

In the mid-18th century, Paris became the centre of an explosion of philosophic and scientific activity challenging traditional doctrines and dogmas. Voltaire and Jean-Jacques Rousseau led the philosophic movement. They argued for a society based upon reason rather than Faith and Catholic doctrine, for a new civil order based on natural law and for science based on experiments and observation. During the Enlightenment, Science was dominated by scientific societies and academies, which had largely replaced universities as centres of scientific research and development; societies and academies were also the backbone of the maturation of the scientific profession. Many women played an essential part in the French Enlightenment, due to the role they played as *salonnières* in Parisian salons, as the contrast to the male *Philosophes*.

Developments in the Industrial Revolution allowed consumer goods to be produced in greater quantities at lower prices, encouraging the spread of books, pamphlets, newspapers and journals. An important development of the Enlightenment was the popularization of science among an increasingly literate population. *Les Philosophes* introduced the public to many scientific theories, most notably through the *Encyclopédie* and the popularization of Newtonianism by Voltaire and Émilie du Châtelet.

Sir Isaac Newton's celebrated *Philosophiae Naturalis Principia Mathematica* was published in Latin and remained inaccessible to readers without education in the classics until Enlightenment writers began to translate and analyse it [21].

The first French introduction to Newtonianism and the *Principia* was the book *Eléments de la philosophie de Newton*, published by Voltaire in 1738. Émilie du Châtelet's translation of the *Principia*, published after her death in 1756, also helped to spread Newton's theories beyond scientific academies and the university.

The Enlightenment played a distinctive, but still rather small, role in the history of Italy. Maria Gaetana Agnesi, born in Milan by a wealthy and literate family, was profoundly interested in mathematics but still unclear about the nature of her possible contribution. Agnesi began by planning a commentary on Guillame de l'Hospital's treatise on curves, to make it more accessible to students. Gradually, however, she came to believe she could work on a much more ambitious project: an introduction to calculus that would guide the beginner from the rudiments of algebra to the new differential and integral techniques. This would be a great work of synthesis, aiming at a clear presentation of materials that were written for specialists, in Latin, French, or German, and published in hard-to-find journals.

Agnesi's geometrical style, which originated in her essentially geometrical understanding of algebra and calculus, was in marked disagreement with leading practitioners.

This explains, among other things, her interest in Newton's fluxions, and the ease with which her work was translated into English for a British audience. At a time when the practice of calculus on the continent was moving away from its immediate geometrical meaning, Agnesi aimed to rediscover those techniques of Cartesian geometry designed to bridge the gap between the two fields.

In the second volume of her treatise *Instituzioni Analitiche* (Analytical Institutions) [1], Maria Gaetana studied many curves, and cycloid in particular, using both points of view: the geometric and the differential method.

2 Cycloid: what was known about it

The problem of finding a tangent to a curve was one of the main reasons, together with the problem of instantaneous speed and acceleration, the problem of maxima and minima and that of the length of the curves and the measurement of areas delimited by curves, which led to the invention of the infinitesimal calculus.

Finding the tangent to a curve was a geometric problem, which had great importance also for its scientific applications: for the optics and for the study of trajectories, topics of extreme interest in the eighteenth century. The same meaning of a tangent was then an open question because the definition of tangent given to conics was inadequate for the more complicated curves, already in use at the time.

Numerous methods were proposed to trace the tangent to a curve. Gilles Personne de Roberval (1602-1675) in his *Traité des indivisibiles* of 1634 (published only in 1693) generalized a method that Archimedes used to find the tangent at every point of the spiral. Roberval imagined the curve as the locus of a point moving under the action of two velocities, horizontal and vertical, and considered as tangent in P the line on which lies the resulting diagonal of these two velocities; Torricelli noted that this method used a principle already enunciated by Galileo and Torricelli himself then used it to find the tangent curves of the $y = x^n$ type. This definition of a tangent was applied to many problems and was remarkable because it linked geometry to dynamics; but it was questionable from the mathematical point of view because, based on physical concepts, it could not apply to those situations that had nothing to do with motion. Therefore, other methods were acquired, including that of Fermat, invented before 1629 and found in his manuscript of 1637 *Methodus ad disquirendam maximam et minimam*. Fermat's method consisted of finding the length of the subtangent to go back up to the point of intersection with the axis and then to the tangent.

The cycloid was one of the curves that played a central role and were used by nearly every mathematician of the seventeenth century as examples for demonstrating new techniques. On 1638, in the book *Discorsi e dimostrazioni matematiche* [11], Galileo Galilei had shown a series of results that can be related to the problem of the brachistochrone. Constructing the tangent of the cycloid dates back to August 1638, when Marin Mersenne received methods from Gilles de Roberval, Pierre de Fermat and René Descartes. Mersenne passed these results along to Galileo, who gave them to his students Torricelli and Viviani, who were able to produce a quadrature. In 1644, Torricelli published this result and others in the first printed work on the cycloid.

In 1657, the Dutch mathematician Christiaan Huygens was thinking about how to make a more accurate clock [16]. Mersenne suggested using a pendulum as a timing device, but Huygens knew that the period of a circular pendulum is not independent of its amplitude and he wrote [25, p. 71]: *In a simple pendulum the swings that are elongated more from the perpendicular are slower than the others. And so in order to correct this defect at first I suspended the pendulum between two curved plates…, which by experiment I learned in what way and how to bend in order to equalize the larger and smaller swings*.

What Huygens did was to place nails in the path of a pendulum made with a flexible cord. The nails altered the path of the bob so that it followed a sequence of circular arcs. By trial and error, he was able to construct a system whose period was independent of its amplitude. To force the bob to travel along a smooth rather than piecewise path, he replaced the nails with a pair of curved plates. Inspired by Pascal's contest, he noticed that the bob of his curved plate pendulum appeared to follow a cycloid. He was able to show that the frequency of an object forced to follow an inverted cycloid is independent of its amplitude.

Thus Huygens proved that the cycloid is the *tautochrone*: the curve for which the time taken by a particle, freely accelerated by gravity, to reach the lowest point on the curve is the same regardless of its starting point. Huygens published this result in 1673, in his book *De Horologio Oscillatorio* [16].

On May 1697, in the *Acta Eruditorum* appeared the solutions of the brachistochrone problem done by Johann and Jacob Bernoulli, together with a note by Leibniz, while Newton had published his solution anonymously on the *Philosophical Transactions*. They proved that the cycloid is the *brachistochrone.*

3 The cycloid in mathematical popularization

Les Philosophes introduced the public to many scientific theories, most notably through the *Encyclopédie* and the popularization of Newtonianism by Voltaire and Émilie du Châtelet. Some works are more formal and include explanations of scientific theories for individuals lacking the educational background to comprehend the original scientific text.

The first significant work that expressed scientific theory and knowledge expressly for the laity, in the vernacular and with the entertainment of readers in mind, was Bernard de Fontenelle's *Conversations on the Plurality of Worlds* (1686) [5]. The book was produced specifically for women with an interest in scientific writing and it inspired a variety of similar works. These popular works were written in a discursive style, which was laid out much more clearly for the reader than the complicated articles, treatises and books published by the academies and scientists. Noted examples of this popular new genre include Francesco Algarotti's *Newtonianism for Ladies* (*Il Newtonianismo per le dame*, 1737), which was the most popular work, and it was translated into English by Elizabeth Carter.

In the Émilie du Châtelet's treatise *The Institutes of Philosophy*, published in Paris in 1738, the Cycloid is presented in Chapter XVIII: *De l'Oscillation des Pendules* in this way:
This curve which is very famous among the Geometry by the number and singularity of its properties, is formed with the revolution of one point of a circle, whose entire circumference subsequently touches a straight line .... The wheels of a carriage turning describe the cycloids in the air.

Émilie du Châtelet, after having described its principal properties, underlines the solution of the brachistochrone problem given by Johann Bernoulli with the dioptric method.

In the *Analytical Institutions, Book II, Example*, Maria Gaetana gave the geometric definition of the cycloid, and then she explained how to obtain, by differential calculation methods, the formula for calculating the subtangent in two different ways, and other properties.

4 The cycloid in Maria Gaetana Agnesi's book

In the book II of the *Instituzioni Analitiche* we find the following definition of the cycloid.

Definition: While the circle DPC revolves uniformly upon the right line AB, beginning at the point A, the point C of its periphery, which at the beginning of the motion fell upon in A, leaves an impression in the plane of its motion that continues till the point C arrives again at the right line AB. It will describe a curve ACB, which, from its generation, is called a *Cycloid* [6].

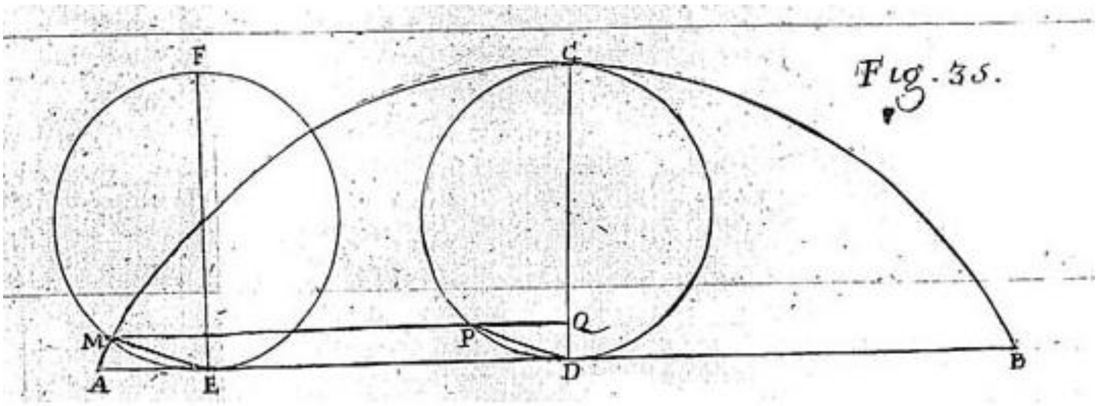

Fig. 1

It will be the ordinary cycloid, when the circle moves upon the right line AB, as that it shall measure out the whole exactly by it's periphery, after that the point C shall have passed from A to B, so that AB may be equal to the periphery of the same circle. It will be a prolonged cycloid when the motion is such, that the right line AB is longer than the periphery of the circle; and a contracted cycloid when the same AB is shorter than the periphery.

From the description of this curve it plainly follows, that, drawing from any point the right line MQ parallel to AB, the intercepted line MP, between the curve and the circle CPD, will have the same ratio to the arc CP as the line AB has to the circumference of the whole circle.

Suppose the generating circle to be in the two positions EMF, DPC; draw the chords ME, PD. Now, because the arcs EM, DP, are equal, the chords EM, DP, will be equal and parallel, and therefore MP = ED. But, by the nature of the curve, it is

$$AE : EM = AD : EMF = AB : EMFE$$

and in the same ratio is also ED : MF. And MF = PC, ED = MP; therefore, it will be

$$MP : PC = AD : EMF = AB : EMFE$$

Therefore, if we call the right line AB = a, the periphery of the generating circle EMFE = b, and any arc or abscissa CP = x, the ordinate PM = y, the equation of the curve of the cycloid will be

$$x = by/a.$$

Having therefore the equation of the curve, in order to find the subtangent, its fluxion will be $dx = (bdy)/a$; and, instead of dx, substituting this value in the formula (ydx/dy, it will be $\mathrm{PT} = (by)/a = x$. Therefore, taking, on the tangent to the circle, PK, (Fig. 2) which is supposed to be drawn, a portion PT equal to the arc of the circle AP, and drawing the right line TM to the point M, it shall be a tangent to the cycloid at the point M.

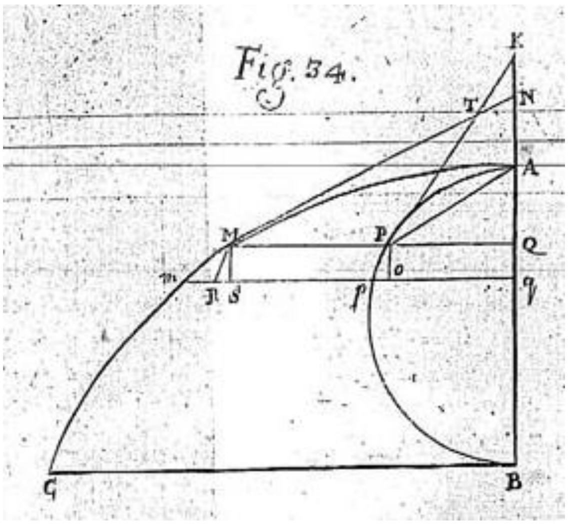

Fig. 2

Now, besides, if the cycloid be the ordinary one; because, in this case, we shall have $b = a$, and therefore $y = x$, it will be PM = PT, and the angle PTM = PMT. But the external angle TPQ is double to the angle TMP, and the angles TPA, APQ are equal, (Euclid, *Elements*, III, 29 and 32), therefore the angle APQ will be equal to the angle TMP, and therefore the tangent MT is parallel to the chord PA.

Without the assistance of the tangent to the curve, we may have the subtangent of the curve AM, taking it in the axis KAB.

Make AQ = x, QP = y, the arc AP = s, QM = z, and let the relation of the arc AP to the ordinate QM be expressed by any equation whatever.

Let qm be infinitely near to QM, and MS parallel to AB. It will be MS = dx, Sm = dz, and the similar triangles mSM, MQN, will give us

$$dz : dx = z : \mathrm{QN} = (zdx)/dz$$

a formula for the subtangent.

Instead of taking for the ordinate QM = z, if we take PM = u; drawing MR parallel to the little arc Pp, it will be

$$m\mathrm{R} = du,\ \mathrm{RS} = pa = dy$$

and therefore

$$m\mathrm{S} = du + dy\,.$$

And the similar triangles *m*SM, MQN, will give us

$$du + dy : x = u + y : \mathrm{QN} \Rightarrow \mathrm{QN} = \frac{(u+y)dx}{du+dy}$$

which is another formula for the subtangent [6].

5 The evolute of the cycloid

In the *Analytical Institutions, Book II, Example VIII*, Maria Gaetana determined, with methods of differential calculus, the evolute of the cycloid.

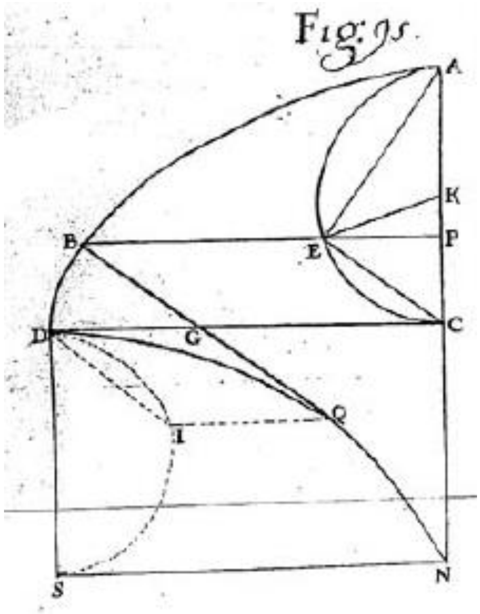

Fig. 3

Let the curve ABD be half of the common cycloid, the equation of which is

$$dy = dx\sqrt{\frac{2a-x}{x}}$$

making AC = *2a,* AP = *x*, PB = *y*. By differencing, and taking *dx* for constant, it will be

$$ddy = \frac{-adx^2}{x\sqrt{2ax-xx}}$$

and substituting these values in the formula for the radius of curvature

$$\frac{(dx^2+dy^2)^{3/2}}{-dxddy}$$

it will be $BQ = 2\sqrt{4aa - 2ax}$.

But the normal $BG = \sqrt{4aa - 2ax}$ which is equal to the chord EC. Therefore, the radius of curvature BQ = 2BG = 2EC.

Making x = 0, to have the radius of curvature at the point A, it will be BQ = AN = 4a, and therefore CN = CA = 2a.

Making x = 2a, the radius of curvature at the point D will be zero, and therefore the evolute begins in D, and terminates in N.

Because the tangent to the cycloid at B is parallel to the chord EA, the normal BQ will be parallel to the chord EC. This supposed, complete the rectangle DCNS and with the diameter DS = CN= AC describe the semicircle DIS and draw the chord DI parallel to BQ or to EC. The angles CDI, DCE, will be equal, and consequently the arcs DI, CE, and their chords. Therefore, DI, GQ are equal and parallel; and drawing IQ, it will be equal and parallel to DG. But, by the property of the cycloid, DG is equal to the arc EC, and therefore to the arc DI. Then the arc DI = IQ, and the semicircle DIS = SN, whence the evolute DQN is the same cycloid, DBA, in an inverted situation.

6 The area subtended by a cycloid arc

In the *Analytical Institutions, Book III, Example VIII*, Maria Gaetana determines, with differential calculation methods, the area subtended by a cycloid arc.

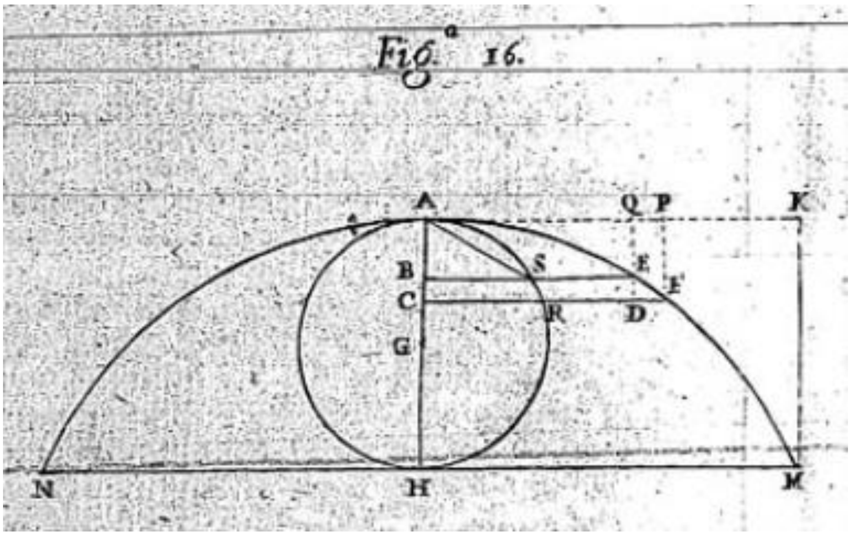

Fig. 4

Let NAM be a cycloid, its generating circle ARH, and make AH = a, AB = x, BC = dx, BE = y, DF = dy.

The equation will be

$$dy = \frac{adx - xdx}{\sqrt{ax - xx}} = \frac{dx\sqrt{a - x}}{\sqrt{x}}.$$

But the little space QEFP is the element of the space AEQ and therefore FP × PQ that is,

$$\frac{xdx\sqrt{a - x}}{\sqrt{x}} = dx\sqrt{ax - xx}$$

will be its formula.

But $\int dx\sqrt{ax - xx}$ is the circular segment ASB; therefore the cycloidal space AEQ will be equal to the correspondent circular space ASB and the whole space AMK will be equal to the semicircle. The rectangle AHMK is quadruple of the semicircle, because it is the product of the semiperiphery into the diameter. Therefore, the space AMH will be triple of the semicircle, and therefore the whole cycloidal space will be triple of the generating circle.

If we would have the space AFC immediately; as the little space FCBE, that is, *ydx*, is its element, and from the equation of the curve we have

$$dy = \frac{dx\sqrt{a - x}}{\sqrt{x}}.$$

Let the *homogeneum comparationis* be reduced into a series, first multiplying the numerator and denominator by $\sqrt{x}$; whence it would be

$$dy = \frac{dx\sqrt{ax - xx}}{x} = \frac{a^{\frac{1}{2}}dx}{x^{\frac{1}{2}}} - \frac{x^{\frac{1}{2}}dx}{2a^{\frac{1}{2}}} - \frac{x^{\frac{3}{2}}dx}{8a^{\frac{3}{2}}} - \frac{x^{\frac{5}{2}}dx}{16a^{\frac{5}{2}}} \cdots$$

and therefore, by integration,

$$\int \frac{dx\sqrt{ax - xx}}{x} = 2a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}dx}{3a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}dx}{20a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}dx}{56a^{\frac{5}{2}}} \cdots$$

whence

$$ydx = 2a^{\frac{1}{2}}x^{\frac{1}{2}}dx - \frac{x^{\frac{3}{2}}dx}{3a^{\frac{1}{2}}}dx - \frac{x^{\frac{5}{2}}dx}{20a^{\frac{3}{2}}}dx - \frac{x^{\frac{7}{2}}dx}{56a^{\frac{5}{2}}}dx \cdots$$

and, by integration, [6]

$$\int ydx = ABE = \frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}dx - \frac{2}{15}\frac{x^{\frac{5}{2}}dx}{a^{\frac{1}{2}}}dx - \frac{x^{\frac{7}{2}}dx}{70a^{\frac{3}{2}}}dx - \frac{x^{\frac{9}{2}}dx}{252a^{\frac{5}{2}}}dx \cdots$$

7 The length of a cycloid arc

Subsequently in the *Analytical Institutions, Book III, Example XXII*, Maria Gaetana determines, with differential calculation methods, the length of the cycloid arc.

Let it be the cycloid of the Example VIII of Quadratures, the equation of which we know to be

$$dy = \frac{dx\sqrt{a-x}}{\sqrt{x}} \quad .$$

Therefore, the formula will be

$$\sqrt{dx^2 + dy^2} = \frac{dx\sqrt{a}}{\sqrt{x}}$$

and therefore, by integration, it will be the arc $\mathrm{EA} = 2\sqrt{ax}$, or the double of the chord AS of the corresponding circular arc AS. Taking $x = a$, AM will be the double of the diameter of the generating circle, and therefore the whole cycloid will be quadruple [6].

8 The witch of Agnesi

When Maria Gaetana was 34 years old, put mathematics aside and turned to charitable work among the poor and sick, she always would dedicate her life. She gave her net worth to the poor and she retired in the Pio Albergo Trivulzio, the poorhouse founded in Milan by the Prince Antonio Tolomeo Trivulzio.

Among the mathematicians, Maria Gaetana is known for having studied a curve, which, in the Anglo-Saxon world, is known as a *witch of Agnesi*. A woman so pious, why should she have given the name "witch" to a curve she had studied?

The name "witch" derives from a mistranslation of the term *versiera* ("versed sine curve") in the original work as *avversiera* ("witch" or "wife of the devil") in an 1801 translation of the *Analytical Institutions* by Cambridge Lucasian Professor of Mathematics John Colson [6].

The name, however, appears for the first time in the *Note al Trattato del Galileo del moto naturalmente accelerato* (Notes to the Galileo's Treaty of the naturally accelerated motion, 1718) written by the famous Italian mathematician Guido Grandi. In this paper, we read that the name *versiera*, in Latin *versoria*, derives from *sinus versus* that is the trigonometric function already appearing in some of the earliest trigonometric tables. The versed sine of an angle equals one minus its cosine [15].

The curve was obtained for the first time from Guido Grandi in his work entitled *Quadratura Circuli et Hyperbolae* (1703), [14]. The attribution to Grandi is also confirmed by the excerpt of the *Exercitatio geometrica in qua agitur de dimensione omnium conicarum sectionum, curvae parabolicae* by Lorenzo Lorenzini (1721) and by Gino Loria *Curve speciali algebriche e trascendenti* (1930), [17].

Grandi in Propositions III and IV of the treaty on *Quadratura Circuli et Hyperbolae* gives its first definition of the *versiera*, followed by interesting properties including:

1) the portion of the plane between the curve and its asymptote, indefinitely extended by the two bands, is equal to four times the generator circle of the curve itself,
2) rotating this portion of the plane around the asymptote, the solid is equivalent to twice that generated in the same rotation by the generating circle.

Grandi finds the equation of the curve in orthogonal Cartesian coordinates, generalizing to curves of which the versiera is a special case. Finally, he applies the versiera to optical considerations. Precisely in this order of ideas, he named the curve *Scala Intensionum*, since the ordinate in each points of the curve represent the intensity of illumination radiated from the source in the same points to which it is applied.

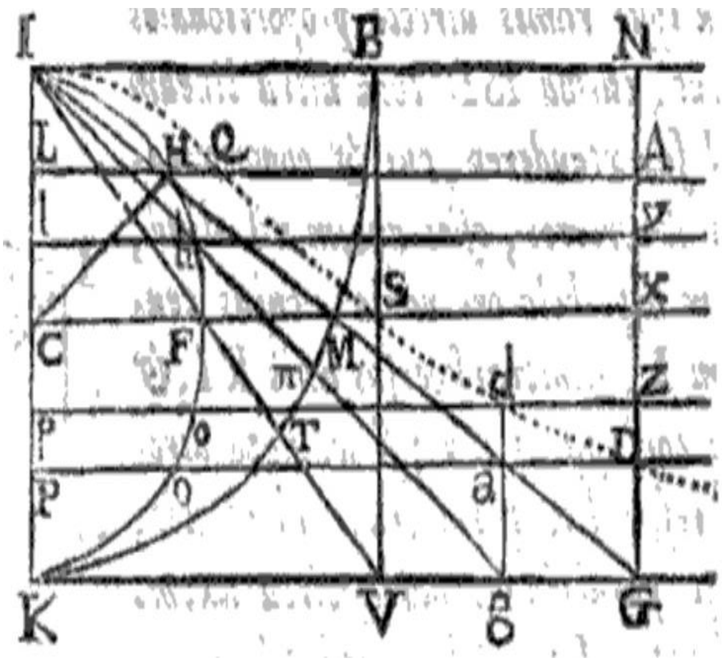

Fig. 5

Maria Gaetana Agnesi defines the versiera in *Book I, Chapter V, Example*, using the geometrical definition given by Grandi in *Note al Trattato del Galileo del moto naturalmente accelerato.*

Given the semicircle ADC of the diameter AC, we consider the point M, out of it, such that, if we draw MB normal to the diameter AC, which will cut the circle in D, we have

$$AB : BD = AC : BM.$$

Because there are infinitely many points M, which satisfy the problem, the locus is asked for.

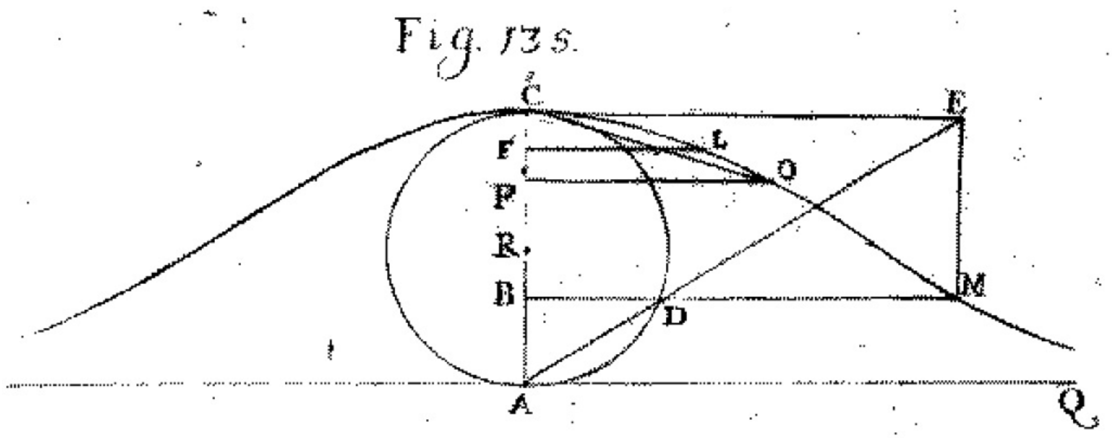

Fig. 6

In *Book I, Problem III*, Maria Gaetana, using simple geometric proportions, deduces the equation of the curve and proves that the curve has a horizontal asymptote. Then she proves, using the definition of convexity, the existence of at least one inflection point. In Example III, Maria Gaetana explains how to draw the curve.

In *Book II, Example II*, Maria Gaetana finds the inflections points, using the differential calculus' method.

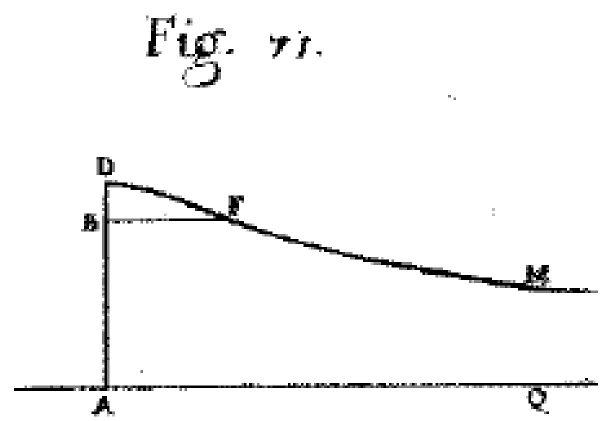

Fig.7

Let the versiera's equation $y = \frac{a\sqrt{ax-xx}}{x} = a\sqrt{\frac{a-x}{x}}$, AB = x, BF = y, AD = a.

We differentiate two times

$$dy = -\frac{aadx}{2x\sqrt{ax-xx}}$$

$$ddy = \frac{(3a^3-4aax)dx}{4x(ax-xx)^{3/2}} \ .$$

For $ddy = 0$ we obtain $3a - 4x = 0$ and so $x = \frac{3}{4}a$; $y = a\sqrt{\frac{1}{3}}$, that is an inflection point.

For $ddy = \infty$ we have $x = \infty$ or $x = 0$, it means that the asymptote AQ and the tangent to D are parallel to the ordinate axis.[1]

Conclusions

When her main book appeared, leading Italian and French mathematicians praised Agnesi's style as clear and effective, but her historiographical fortune declined rapidly towards the end of the century and never quite recovered.

The biographies of Agnesi rehashed a very limited amount of information, mostly anecdotal, derived from a first biography published by Antonio Frisi in 1799 [9].

Nineteenth and twentieth-century historians did refer to Agnesi as a heroine of the Enlightenment, but always bearing in mind the necessary limitations of her gender and therefore of her technical and conceptual accomplishments. Indeed, the belief that the practice of mathematics is essentially gendered is not as distant as some of us might like to think. One should just remember that, in 2005, Larry Summers, then president of Harvard University, speculated that behind the gender gap in top science and engineering jobs there might be *issues of intrinsic aptitude*.

In 1989, Clifford Truesdell published the most in-depth study of Agnesi's scientific work in which he concluded that Agnesi must be appreciated simply because she was a woman engaged in mathematics at a time when this activity was entirely dominated by men, but there is nothing in her work that justifies special attention.

Maria Gaetana Agnesi was famous in her time, mainly as an isolated, unique, female prodigy, a marvel first of precocious learning, later of Catholic piety. Misty fame still clings to her memory, long nourished by the occasional curiosity of mathematicians and now revived by feminists. Whether or not she was as a mathematician important in any sense - discoverer, propagator, teacher - can be determined, but her brief and intense devotion to elementary mathematics cannot be separated from her strange circumstances and strange life, and as her story unfolds, a certain unity, at first unexpected, can be perceived in it [24].

Truesdell's judgment on the *Instituzioni* is in clear continuity with remarks of Gino Loria; in his opinion, indeed, this book is a mere work of popularization that lacks originality, and therefore historical interest.

The work does not excel in originality; ... but it is distinguished by clarity and rigor of style and by numerous and interesting applications; consequently... it was judged so favourably as to arouse the enthusiasm of some . . . [17].

[1] All the figures in this paper are reproductions of the original figures in the cited works. We have kept the style of Agnesi's proofs as much as possible.

The opinions on the mathematical work of Maria Gaetana Agnesi is still controversial, mainly because she did not discover any new analytical properties.

Luigi Pepe describes Agnesi's book as an *exposition by examples rather than by theory* [23].

However, the same could be said of John Bernoulli's integral calculus, written in 1691 and 1692, published in 1742. All of Euler's books are full of examples.

The style and content of the *Instituzioni* provide us with precious indications about Agnesi's intentions and goals. The book presents indeed some distinctive features when compared to contemporary treatises. It looks like a hybrid of different mathematical traditions, namely the Leibnizian-Bernoullian and the Newtonian. It is written in Leibnizian algebraic notation, but the thinking behind it seems always genuinely geometrical, as was proper in the Newtonian tradition. It is not a coincidence that the *Instituzioni* would attract the interest of some British scholars of the nineteenth century, at a time of bitter disputes about the respective merits of the two competing approaches [20].

Many elementary textbooks today, especially those used in courses for engineers, physicists, and economists, are of that kind as far as concerns mathematical thought, but they usually offer redeeming examples of how to apply calculus to problems suggested by natural or fancied phenomena. In our opinion, Maria Gaetana makes a more general vision of problems and contributes to favour the mathematical abstraction [19].

We can propose these arguments or problems by means of laboratory instruments, flipped classroom techniques, or by didactical methods and we think that the media are different but the meaning is the same. According to some scholars, the abstract approach, nowadays we use, can be didactically not very effective for the beginner student. They think that an intuition of infinitesimals can be oriented to lead to mathematical concepts. From this point of view, we present the way in which Maria Gaetana Agnesi presented the cycloid, a traditional curve that nearly every mathematician used as example for demonstrating the power of the differential calculus techniques.

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Reinterpreting Pablo Picasso's paintings using Bézier curves

Cristina Păcurar

Abstrakt
Predložený článok pozostáva z dvoch častí venovaných reinterpretácii diel Pabla Picassa pomocou Bézierovych kriviek. V prvej časti je cieľom použiť Bézierove krivky na reprodukciu slávnej kresby Pabla Picassa, ktorá bola vytvorená ako čiara jedným ťahom. Uvedená je aj ďalšia reprezentácia tejto kresby, ktorá sa javí ako typickejšia pre umelecké hnutie kubizmus. V druhej časti analyzujeme portrét Dory Maar a pokúsime sa nájsť iný vzhľad tejto maľby procesom inverzným k procesu použitému v prvej časti článku.

Kľúčové slová: Pablo Picasso, prosté umenie, kresby, Bézierove krivky, Tête de femme, Dora Maar, kubizmus, Bernštejnove polynómy.

Abstract
The present paper is composed of two parts concerning Pablo Picasso's artworks reinterpreted through Bézier curves. For the first part, we aim to reproduce a famous line drawing of Pablo Picasso with the use of Bézier curves. Moreover, we also give another representation for the drawing from a point of view more specific to the Cubism artistic movement. In the second part, we analyze one of the portraits of Dora Maar and try to find another appearance for the painting through an inverse process from the one applied in the first part.

Key words: Pablo Picasso, low-complexity art, drawings, Bézier curves, Tête de femme, Dora Maar, Cubism, Bernstein polynomials.

1 Bézier curves

Bézier curves are parametric curves which are widely spread in many areas and have a variety of applications in many different fields. The curves are named after Pierre Bézier, who published his work about the curves in 1962.

Bézier curves are based on *Bernstein polynomials*. They are in fact defined by a set of points, called control points, from which the first and the last one represent the starting point and the ending point of the curve. The other intermediate points are usually not on the curve.

A linear Bézier curve is a line segment defined by the starting point P_0 and ending point P_1 and linear Bernstein polynomials:

$$B(t) = (1-t)P_0 + tP_1\,, \qquad t \in [1,0]\,.$$

Moving on to quadratic Bézier curves, their representation is given by the second degree Bernstein polynomials and 3 consecutive points P_0, P_1 and P_2:

$$B(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2\,, \qquad t \in [1,0]\,.$$

The discussion can be generalised to the Bézier curves of degree n determined by $n + 1$ control points and Bernstein polynomials of degree n defined by formula:

$$Be_{in}(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i \,, \qquad t \in [1,0] \,.$$

Bézier curves have been widely studied and are a powerful and highly important tool in geometric modelling of approximation curves, which increases their usefulness in computer graphics, animations and other numerous fields.

2 Pablo Picasso

Pablo Ruiz Picasso was born on 25th October 1881 in Malaga, Spain, but he spent most of his life in France. He was a multilateral Spanish plastic artist, with over eighty years ofprodigious activity as a painter, a sculptor, a poet, a playwright and a stage designer. Picasso was one of the most influential figures of the 20th century.

He is one of the founders of the artistic movement called *Cubism*. He changed the course of art with his controversial oil painting from 1907, *Les Demoiselles d'Avignon*, as it "was only a first step towards a complete revolution in the world of art; it heralded the beginning of cubism" [4]. Cubism brought to light a painting of a dissected subject, which is put under a magnifier glass not only as a tangible presence, but much more as a keeper of unknown secrets in its essence. Thus, the resulting canvas in Cubism is a mixture of the subject's essence, which may reduce the real appearance of the object to even eliminating it completely, and the author's feelings toward everything the subject represents.

One of Picasso's trademarks is the importance of lines throughout his artworks, which are called *the backbone to his art* by Christopher Lloyd [8]. The present paper analyses some of the artist's line drawings, which have an undeniable significance and an incontestable beauty. Besides the drawings, the paper approaches one of the many paintings portraying one of Picasso's muse, Dora Maar. Despite the fact that the painting in question is not a line drawing, the importance of the lines in this artwork is prominent.

3 Picasso's drawings

Picasso has made some experiences during his life with some drawings that strike the viewer with their simplicity, which actually hides a great amount of complexity. In [7], an impressive collection of Picasso's drawings can be found. The drawings from Fig. 1 and Fig. 2 are just some examples of the incredible one-line drawings that can be found in [7].

Picasso's inspiration for line drawing came from various sources, as the painter chose numerous subjects, including animals, mythology and even portraits or still life. Despite this significant variety of subjects, the connecting element of all drawings remains the striking simplicity which paradoxically manages to transmit a lot.

One of his simple single line drawings pictures is a dog, which can be seen in Fig. 2a). This dog was actually the artist's dog, a dachshund named *Lump*.

The drawing of Picasso's own dog fascinated the viewers and suggested the idea of reproducing this piece of art. In [6], a reproduction of the *Dog* based on Bézier curves is realized.

a) Squirrel b) Sparrow c) Camel

Fig. 1. One line drawings

a) Dog b) Owl c) Grasshopper

Fig. 2. One line drawings

4 The Sparrow - Pablo Picasso

One of Picasso's drawings depicts a sparrow, Fig. 1b). The sparrow is just a single continuous line, leaving a hasty viewer with the false impression that the sketch has remained unfinished. Despite this, the drawing manages to perfectly capture not only the essence of the subject, but also its movement, as the sparrow seems to be taking a walk just across the spectator's horizon.

Looking at this simplistic, yet tremendously powerful drawing while taking into consideration the way Bézier curves are characterized (and are obtained), the idea of approaching Picasso's *Sparrow* with those type of curves seems to be achievable.

An important advantage in the pursue to obtain Picasso's *Sparrow* is the fact that Python programming language offers a package specialized in *Bézier* curves, called *bezier*. This represents a valuable support in reproducing Picasso's piece of art with the help of Bézier curves. The starting point for the *Sparrow* is to determine all the control points that will define the curve.

Firstly, we take as initial and, respectively, final point all the inflexion points that can be spotted on the drawing. The next step is to choose the minimum number of additional control points

	Start Point	1st Control Point	2nd Control Point	End Point
Curve 1	(1.66, 1.41)	(2.82, 0.4)	(5.4, 4.56)	(6.9, 4.87)
Curve 2	(6.9, 4.87)	(5.3, 10.84)	(2.07, 8.77)	(2.67, 6.9)
Curve 3	(2.67, 6.9)	(3.15, 7.76)	(3.8, 6.96)	(2.67, 6.9)
Curve 4	(2.67, 6.9)	-	-	(2.54, 6.58)
Curve 5	(2.54, 6.58)	(3.03, 6.93)	(3.35, 6.15)	(3.02, 4.56)
Curve 6	(3.02, 4.56)	(3.45, 3.33)	(4, 1.92)	(4.5, 0.84)
Curve 7	(4.5, 0.84)	(4.13, 0.86)	(3.73, 0.84)	(3.33, 0.8)

Tab. 1. Control points for the *Sparrow*

which will model the line, in order to obtain the desired curve of the smallest degree. The control points which have been considered are listed in Tab. 1. As it is noticeable from Tab. 1, except for Curve 4, which is a linear Bézier curve, the other six ones are cubic Bézier curves determined by the Bernstein polynomials of order three and 4 control points, and represented by the following formula

$$B(t) = (1-t)^3P_0 + 3t(1-t)^2P_1 + 3t^2(1-t)P_2 + t^3P_3, \qquad t \in [0,1].$$

In order to manipulate the control points, the *numpy* Python package is required. We introduce the control points from Tab. 1 as an array, using the built-in numpy function *asfortranarray*, which converts the input to a n-dimensional array (ndarray) with column major memory order. The values are hold into variables named *nodes* with the indicative number of the corresponding curve. The Bézier package's feature *Curve* generates the corresponding Bézier curve based on the *nodes* array and the specification of the curve's degree. For example, the first curve is generated as shown in Example 1.

Example 1

```
import numpy as np
import bezier
nodes1=np.asfortranarray([
#x coordinate  for the control points
[6.9,     #starting point for curve1
5.4,      #2nd control point
2.82,     #3rd control point
1.66],    #end control point
#ycoordinate
[4.87,    #starting point
4.56,     #2nd control point
0.4,      #3rd control point
1.41],    #end control point])
#generate bezier curve from nodes1, with degree4
curve1 = bezier.Curve(nodes1, degree=4)
```

After creating all the Bézier curves in the same way, we generate the xy axes system in which the curves are to be plotted. The *bezier* package also benefits of its own means of creating the plot, through the built-in package *plot*. Thus, using only seven Bézier curves, we can obtain a true reproduction of Picasso's *Sparrow*. The copy of the artist's piece of art is illustrated in Fig. 3, where each of the distinct Bézier curves used is contoured in a different color for a better outlook of the result.

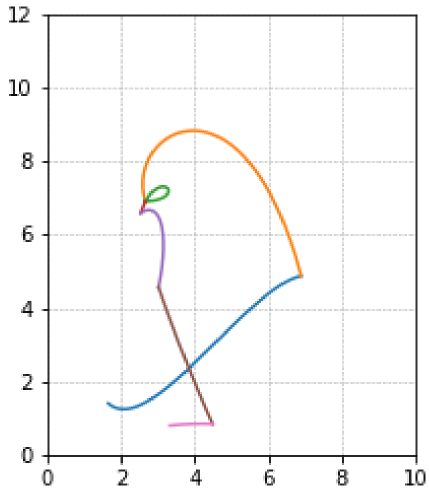
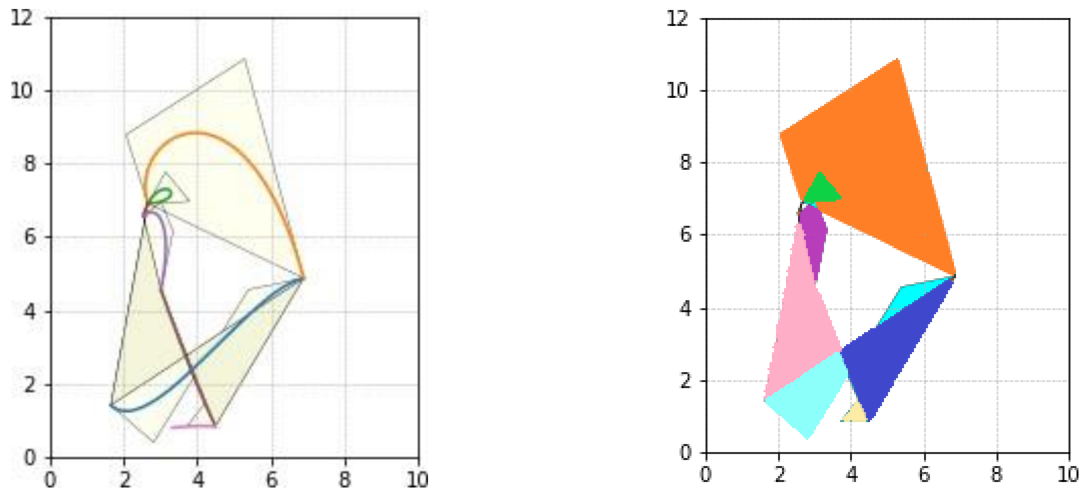


Fig. 3. Python reproduction of the *Sparrow* with Bézier Curves

The process of redrawing the *Sparrow* involves 26 control points, from which 7 are repeated. However, in the final image, only those 7 points are actually observable. Thus, bearing in mind some of the traits specific to Cubist artworks, we aim to reinterpret the initial model of the *Sparrow*. To do so, we represent all the control points from Tab. 1 in Fig. 4a) and display them along the quadrilaterals obtained from joining them.



a) *Sparrow's* reproduction with control points

b) Quadrilaterals obtained by joining control points for the *Sparrow*

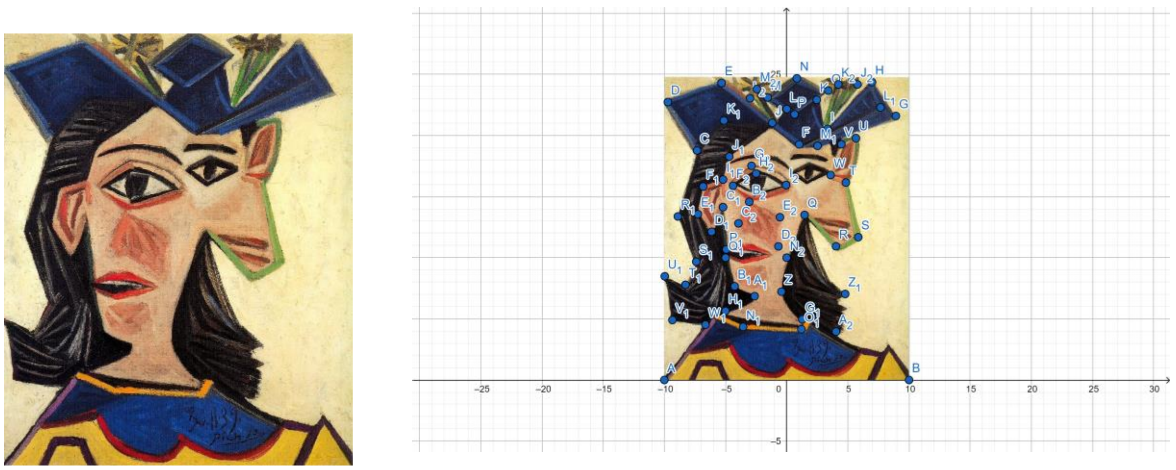
Fig. 4. The *Sparrow*

Fig. 4b ignores the initial model, the *Sparrow* and consists of an independent new representation, from a point of view more specific to the Cubism artistic movement. The sharp edges image which is obtained shows a different face of the beautiful inspiring smooth single line that is Picasso's *Sparrow*.

5 Tête de femme (Dora Maar)

The process of decreasing the complexity of its subjects, which Picasso experimented with, has as a result the beautiful masterpiece drawings presented in sections 3 and 4. However, looking at Picasso's paintings, one may more often find unusually complicated pieces of art, than some simple and faithful to the reality drawings, like the *Sparrow*.

Naturally, the question is arising, how to come to some more simplistic, drawing like image, from a piece of art that has been complicated to look less like its original model. In this section we try to reach an image which is simpler and closer to the reality of the initial subject, starting from a distorted model pictured in one of Picasso's paintings.



a) *Tête de femme* (Dora Maar) b) *Tête de femme* with control points

Fig. 5. *Tête de femme*

During his life, Picasso has repeatedly painted the same subject in different styles or just changing a slightly bit the perspective. This is also the case with Dora Maar. Henriette Theodora Markovitch, who is known as Dora Maar, was a photographer and painter. She met Picasso in 1936 and soon became his muse and lover for almost a decade. She is well known for having photographed Picasso's *Guernica* while it was being painted. During their time together, Dora Maar has been portrayed in all manners and styles that Picasso has experienced by that time in his life. Among those numerous portraits of her, a painting from 1941, shown in Fig. 5 a strikes the viewer's eye with its perfectly sharp edges. The face is only composed of polygons, from which the majority are quadrilaterals.

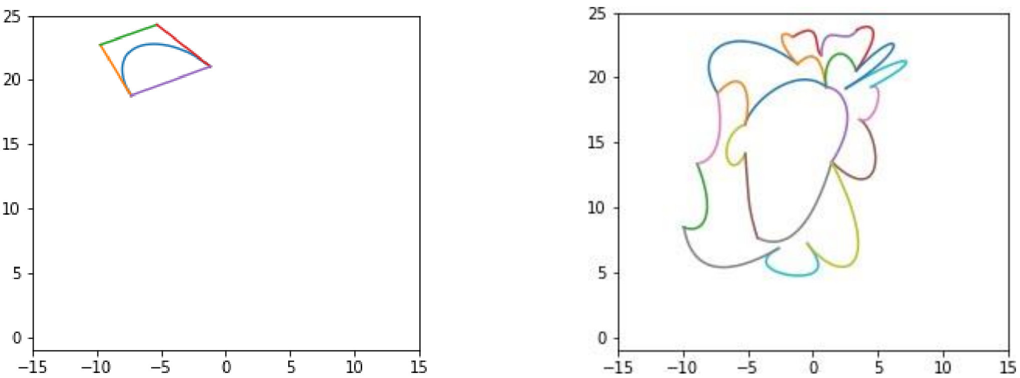
Bearing in mind how a Bézier curve is obtained, and keeping a visual representation of the quadrilateral obtained from the control points needed for a cubic Bézier curve, the idea of recomposing an image closer to the real model seems feasible. In those circumstances, the portrait of Dora Maar may be subjected to an inverse procedure to the one approached at the end of section 4.

The first step, just as in the previous sections, is to establish the control points. In order to do so, we mark each of the vertices of the polygons from the painting and consider them control points.

Fig. 5b) has all the vertices of the initial painting marked with a corresponding point. The purpose is to maintain those exact points and try to create Bézier curves of an appropriate degree based on the sharp initial image.

For example, the upper left corner, which seems to be part of the blue hat of the subject, is the quadrilateral *CDEF*. In Fig. 6a), we can observe that taking as starting point *C*, 1st control point *D*, 2nd control point *E* and end point *F* , we obtain a Bézier curve which will be the dark blue curve in the upper left corner of the hat in Fig. 6b).

Repeating the process of marking points for a polygon with n vertices, converting it to a n-th degree Bézier curve and plotting it on the same canvas, we obtain the full image in Fig. 6b). The contour obtained resembles the contour of a face from the side, with the right ear showing a bit, with curled hair that frames the head, and an interesting hat on top of it. Of course, the obtained image can be modified to resemble more a face. However, the fact that following exactly the vertices of the initial painting lead us to a model that may be a contour of the initial model, makes the process of using Bézier curves in understanding paintings a valuable tool.



a) Detail of *Tête de femme* (Dora Maar)   b) *Tête de femme* with Bézier curves

Fig. 6. Bézier curves reinterpretation

For that matter, Fig. 6b) depicts the result of combining 21 Bézier curves obtained from the control points marked on Fig. 5b).

Although the image obtained lacks the most important part, the facial features, one can make an imagination exercise and add the expressive eyes from the original image to the obtained contour. The substance of the subject seems to be always laying inside the eyes. Thus, the eyes of Dora Maar could not be represented with Bézier curves without loosing their meaning. Despite the fact that the eyes obey somehow the sharp edges of the entire painting, they are the closest to reality. Modifying their proportions a slightly bit, the eyes follow the real model and remain a portal to the essence of the subject.

6 Conclusions

The process of simplifying the art while reducing it to its substance without losing any of the illusions and emotions it initially creates is one of Picasso's distinctive traits. This is strongly related to its inverse process, which makes the art more complex, adding its hidden mathematics to the front desk, ready to face the viewer. The painting *Tête de Femme* (Dora Maar) is just one example of Picasso's paintings which reveals its secrets to the viewer while hiding its original reality. Besides being connected by their complementarity, the two processes strike in similarity while referring to the opposite means they use in order to reproduce the same essence.

Bézier curves are not only a means of reproducing Picasso's drawings, but they are a significant tool to reinterpret some famous artworks. The curves might be an instrument to shift from drawings and paintings that are faithful to the reality to some examples of art more specific to Cubism, as we showed in section 4 and also act in the aid of a counter process, of revealing a more reality-like model from the initial painting, like in section 5.

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Shape distribution approach to measure similarity of triangular meshes

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Abstrakt

Určení míry podobnosti mezi dvěma 3D objekty patří k vysoce náročným problémům počítačové grafiky. V tomto článku jsou k výpočtu míry podobnosti trojúhelníkových sítí, které byly získány optickým skenováním, použita tvarová rozdělení tří různých tvarových funkcí. Vyhodnocení založené na analýze systému měření (MSA – measurement system analysis), což je statistická metoda běžně užívaná v systému řízení jakosti, je použito pro posouzení způsobilosti jednotlivých tvarových rozdělení.

Klíčová slova: trojúhelníková síť, tvarová funkce, tvarové rozdělení, MSA

Abstract

The need to measure a similarity between two 3D objects belongs to highly challenging problems in computer graphics. In this paper, a shape distributions approach based on three different shape functions is used to compute a similarity measure of triangular meshes obtained by optical scanning. The evaluation based on a measurement system analysis (MSA), a statistical method commonly used in quality management systems, is applied to assess the capability of individual shape distributions.

Key words: triangular mesh, shape function, shape distribution, MSA

A measurement system analysis (MSA) is defined as an experimental and statistical method to identify the sources of variation in a measurement process and determine the value of these variations [1]. Based on calculation of variance range of measurement repeated several times, it is possible to assess statistical competence or incompetence of the most important variation sources, i.e. operators and measurement equipment. In this paper, the MSA approach is modified, so that the capability of three different shape distributions to calculate similarity measure of triangular meshes is assessed. Shape functions that measure simple geometrical features on a 3D model (distance, area, volume, etc.) are used in shape distributions approach to simplify solution of shape matching or shape-based recognition problem [2], [3]. If the shape distributions are used, two probability distributions sampled from a suitable shape function are compared instead of traditional shape matching methods (e.g. parametrisation, feature correspondence and model fitting). Originally, the shape distribution is calculated for a sufficiently large number of random sample points of the surface [2]. In this paper, where the shape distributions approach is applied on triangular meshes, all the mesh vertices are considered to be the random sample points. The meshes were obtained by repeated optical scanning of the same ball-bar standard by means of three different types of scanners. The shape distributions of all the meshes were calculated and compared with the shape distributions of the nominal mesh that was generated on the theoretical CAD model of the standard.

The paper is organised as follows. All the triangular meshes compared here were obtained by optical scanning of ball-bar standard commonly used for calibration of optical scanners. The scanned ball-bar standard is shortly introduced in Section 1. Definition of shape functions based on measurement of distances of specific points and areas of specific triangles is described in Section 2. The application of shape functions in shape distribution approach when comparing a similarity of triangular meshes is given in this section, too. Basic ideas of MSA method, an experimental and statistical method to determine the amount of variation within a measurement process caused by individual factors, are explained in Section 3. The modification of MSA method to assess an ability of individual shape distributions in similarity measurement of triangular meshes is also described in this section. Conclusions on the ability of individual shape distributions described in this paper to measure triangular meshes similarity are summarized in Section 4.

1 Ball-bar standard

The ball-bar standard with two precise spheres connected by a cylinder is commonly used for calibration of optical scanners. The standard was scanned by three different portable handheld optical scanners *S1*, *S2* and *S3* separately five times so that fifteen triangular meshes in general position with respect to the coordinate system were obtained. An example of a part of one triangular mesh is shown in Fig. 1. Obviously, the scanned meshes are imperfect because they contain false reflections. These reflections have to be removed before shape distributions approach application. Therefore, all the meshes were pre-processed firstly, i.e. two spheres were calculated by least squares fitting, the meshes were aligned with respect to the coordinate system (the centre of straight line segment with endpoints at centres C_1 and C_2 of the spheres lies at origin O and $C_1C_2 \subset x$). Secondly, all inappropriate parts of scanned handles and false reflections (see Fig. 1) that could distort the results were trimmed, see Fig. 2a). Finally, a triangular mesh of the nominal CAD model of the standard was generated (nominal mesh). The position of the nominal CAD model with respect to the coordinate system is identical to the position of the aligned meshes, see Fig. 2b).

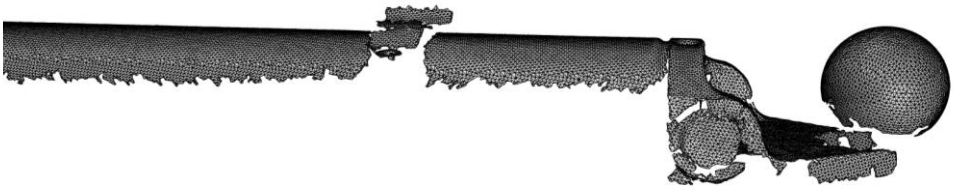

Fig. 1. Part of a triangular mesh of the standard before pre-processing

2 Shape distributions

The shape distributions approach to calculate a similarity measure of triangular meshes consists in the following steps: shape function definition, shape distributions construction and shape distributions comparison.

2.1 Shape functions

In this study, the modifications of $D1$, $D2$ and $D3$ shape functions introduced in [2] are used. Originally, all the three shape functions are defined for points, randomly sampled on the boundary surface of a 3D model, and measure non-oriented distances. Here, the random

sampling is represented by $n+1$ mesh vertices $V_i = (x_{V_i}, y_{V_i}, z_{V_i})$, $i = 0,1,\dots,n$, and, due to the symmetry of the scanned object, oriented distances are taken into consideration.
In particular, the $D1$ shape function measures the oriented distance of a mesh vertex from the origin. The orientation is given by algebraic sign of x-coordinate of the mesh vertex. $D1$ shape function is calculated by

$$f_i = \operatorname{sign}(x_{V_i})\sqrt{x_{V_i}^2 + y_{V_i}^2 + z_{V_i}^2}, \quad i = 0,1,\dots,n. \tag{1}$$

$D2$ shape function measures the oriented distance between every two mesh vertices. The algebraic sign of x-coordinate of midpoint $S_i = (x_{S_i}, y_{S_i}, z_{S_i})$, $i = 0,1,\dots,n$ of the line segment defined by the two mesh vertices determines the orientation. $D2$ shape function is given by

$$\hat{f}_i = \operatorname{sign}(x_{S_i})\|\boldsymbol{a}_i\|, \; i = 0,1,\dots,n, \tag{2}$$

where $\|\boldsymbol{a}_i\|$ is the magnitude of a vector defined by two mesh vertices.
$D3$ shape function measures the square root of oriented triangle area. The triangle vertices are represented by the origin and two mesh vertices. The orientation is determined by the algebraic sign of x-coordinate of triangle centroid $T_i = (x_{T_i}, y_{T_i}, z_{t_i}), i = 0,1,\dots,n$. $D3$ shape function is given by the formula

$$\tilde{f}_i = \operatorname{sign}(x_{T_i})\sqrt{\frac{\|\boldsymbol{u}_i \times \boldsymbol{v}_i\|}{2}}, \; i = 0,1,\dots,n, \tag{3}$$

where $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are the position vectors of mesh vertices.

2.2 Shape distributions

To construct a shape distribution, it is necessary to construct a frequency histogram, i.e. it has to be determined how many values f_i, $\hat{f}_i$ and $\tilde{f}_i$ fall into each of k fixed sized classes. The frequency is normalized by number of mesh vertices $n+1$ to eliminate the influence of different number of meshes vertices. An example of three histograms constructed for $D1$, $D2$ and $D3$ shape functions of nominal mesh is shown in Fig. 2 c), d) and e) in the given order.

2.3 Shape distributions comparison

The shape distributions are represented by the relative frequency histograms. The similarity measure between two shape distributions is based on Minkowski L_1 norm. In this study, the shape distribution of nominal mesh is considered the reference distribution. Consequently, the measurand in comparison is given by Minkowski L_1 norm calculated by formula

$$D(f, g) = \sum_{i=1}^{k} |F_i - G_i|, \tag{4}$$

where F_i represents $D1$ relative frequency of the scanned mesh and G_i represents $D1$ relative frequency of the nominal mesh. Similarly, L_1 norm for $D(\hat{F}, \hat{G}) = \sum_{i=1}^{k} \left|\hat{F}_i - \hat{G}_i\right|$, and

$D(\tilde{F},\tilde{G})=\sum_{i=1}^{k}\left|\tilde{F}_i-\tilde{G}_i\right|$, can be obtained, where $\hat{F}_i$ and $\tilde{F}_i$ represents $D2$ and $D3$ relative frequency of the scanned mesh and $\hat{G}_i$ and $\tilde{G}_i$ represents $D2$ and $D3$ relative frequency of the

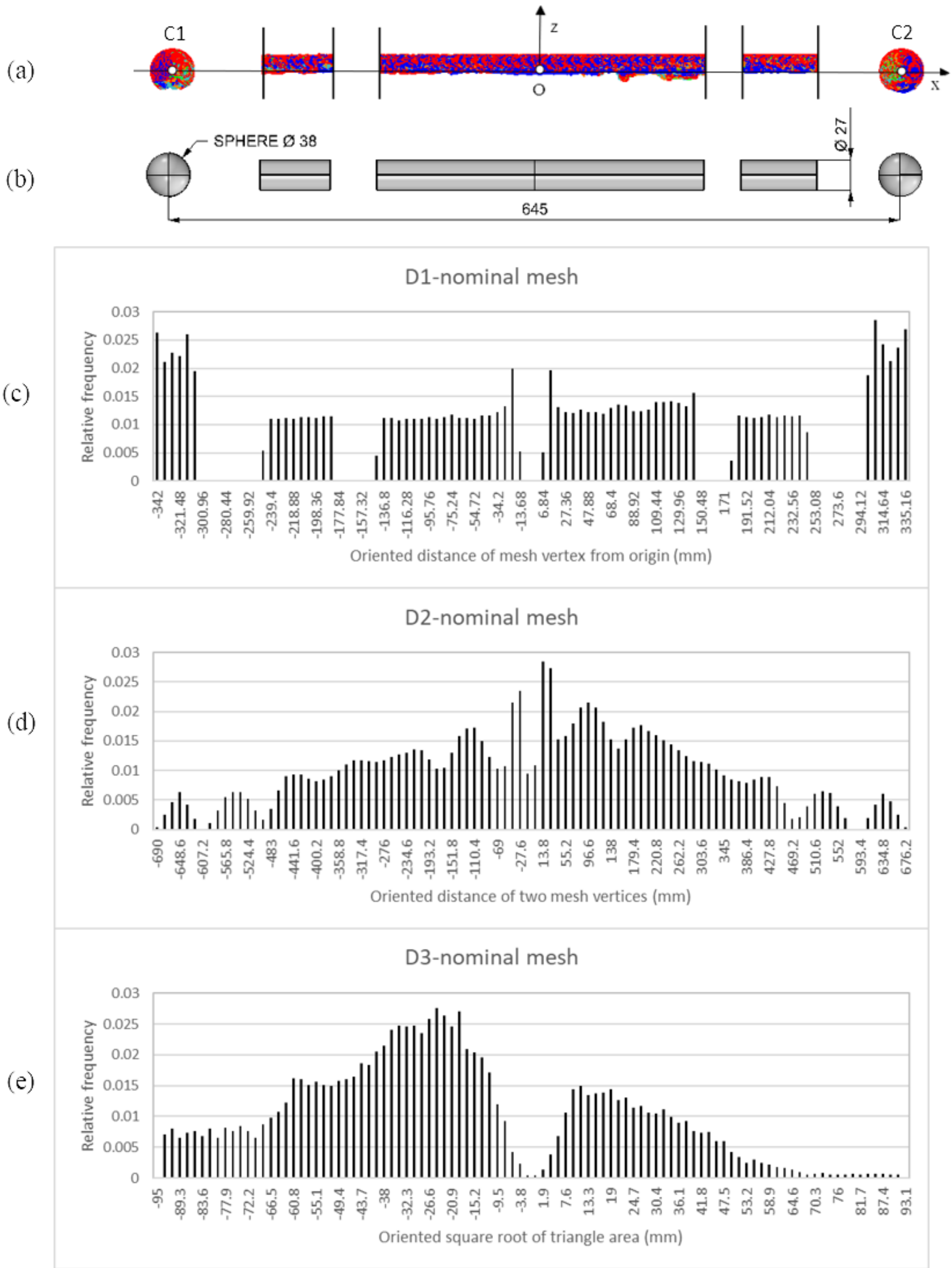

Fig. 2. Relation among triangular meshes, nominal CAD model and shape distributions:
a) Aligned meshes,

b) Nominal CAD model,
c) *D*1 relative frequency histogram of nominal mesh,
d) *D*2 relative frequency histogram of nominal mesh,
e) *D*3 relative frequency histogram of nominal mesh.

nominal mesh in the given order. An example of shape distributions comparing the meshes scanned by scanner *S*1 is drawn in Fig. 3 (individual meshes are designated by *M*1, *M*2, …, *M*5, the nominal mesh by *M*0). Numerical values of L_1 norms for all comparisons are highlighted by grey colour in Tab. 1.

Considering that all meshes are compared to the same reference model, it can be expected that the smaller L_1 norm, the more accurately the mesh is scanned. Simultaneously, it can be assumed that the shape of the scanned mesh will be reflected better by the shape function, whose mutual deviations of relative frequency histograms will be minimal. Although not all measurements are shown in the Fig. 3, there is an obvious shape resemblance of histograms, especially for *D*1 and *D*2 shape distributions. *D*3 shape distribution shows the greatest mutual deviations between the individual histograms. To assess all these aspects, the method of MSA is used.

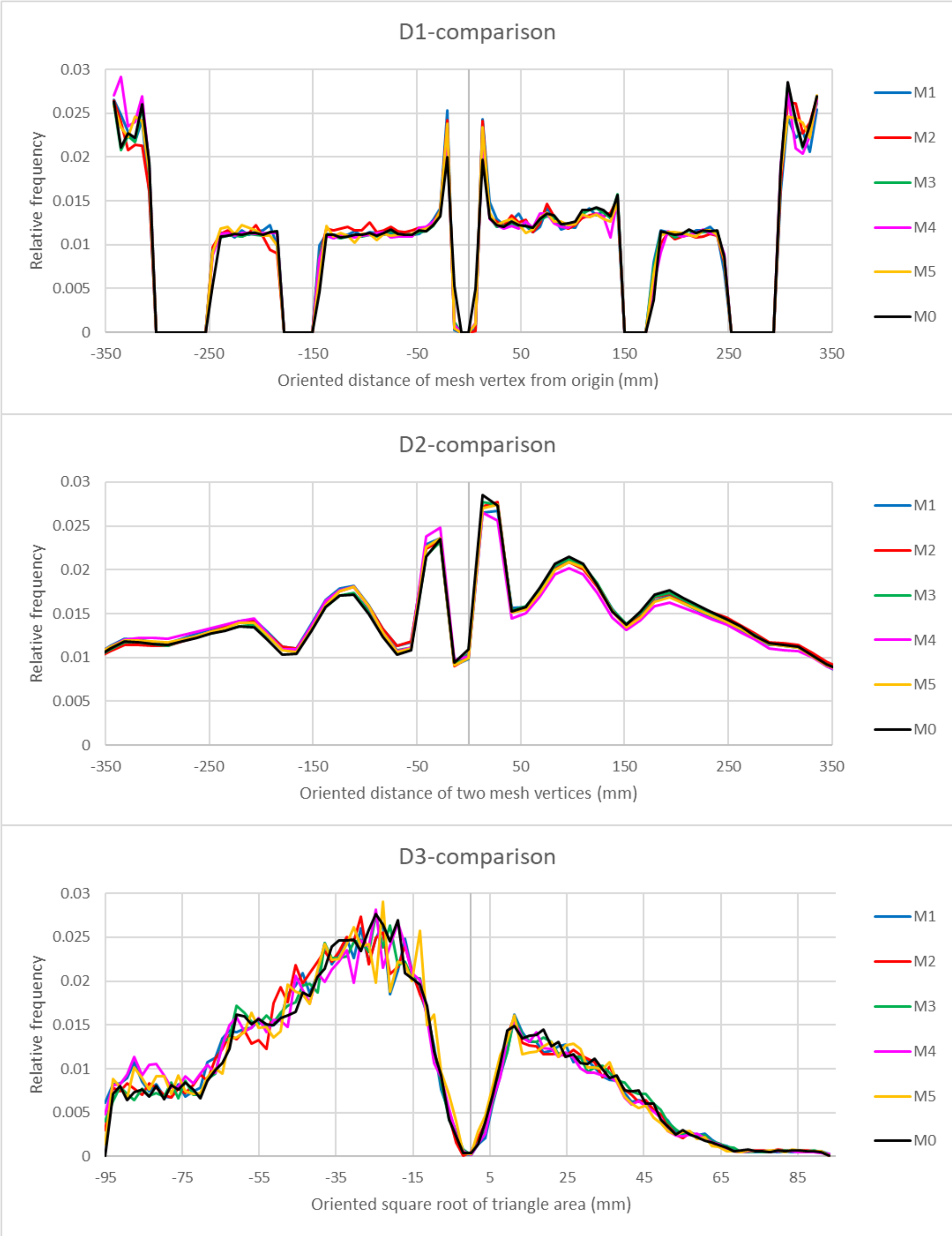

Fig. 3. Comparison of shape distributions for triangular meshes obtained by scanner *S*1

3 MSA

MSA is defined as an experimental and statistical method of determining the amount of variation within a measurement process. This method is usually performed using *h* appraisers (operators) measuring *r* parts *m* times. There are two main components of variation – repeatability (one appraiser measures the same part using the same measuring equipment more than one time) and reproducibility (variation in the average of the measurements made by the different appraisers when measuring the same part). Here, the MSA method is modified as follows: the *D*1, *D*2 and *D*3 shape distributions represent the appraisers ($h = 3$), the *M*1 to *M*5 scans represent the parts ($m = 5$) and the *S*1, *S*2 and *S*3 scanners represent the repeated measurement ($r = 3$). The aim is to determine whether the considered shape distributions are statistically competent or incompetent, which means whether or not they can be recommended for similarity of triangular meshes measurement.

The first step is to calculate the variation range for each measurement for each shape function (see Tab. 1)

$$R_{ij} = \max_k x_{ijk} - \min_k x_{ijk} \tag{5}$$

where x_{ijk} is the measured value (L_1 norms in grey cells), $i = 1,2,3$ indicates the shape distributions $D1$, $D2$, $D3$, $j = 1,2,3$ indicates the scanners $S1$, $S2$, $S3$ and $k = 1,2,3,4,5$ indicates the order of the scan. After that, the following values are calculated: the average variance range of repeated scans achieved by the individual shape distribution (see Tab. 1)

$$\bar{R}_i = \frac{1}{r}\sum_{j=1}^{r} R_{ij} \tag{6}$$

the average variance range of repeated scans achieved by all shape distributions

$$\bar{\bar{R}} = \frac{1}{h}\sum_{i=1}^{h} \bar{R}_i = 0.1686 \tag{7}$$

the control zone given by upper control limit

$$UCL = D_4\bar{\bar{R}} = 0.4349 \tag{8}$$

and lower control limit

$$LCL = D_3\bar{\bar{R}} = 0. \tag{9}$$

Note that the table constants $D_3 = 0$ and $D_4 = 2.58$ for $r = 3$.
Finally, based on the range chart drawn in Fig. 4, the statistical competence or incompetence of the process is assessed. If all values of variance range R_{ij} lie within the control limits

$$R_{ij} \in [LCL, UCL] \tag{10}$$

for each shape distribution, the process is statistically competent. Otherwise, the shape distribution whose values exceed the control limits is statistically incompetent and cannot be recommended for similarity measurement. Obviously, $D3$ shape distribution is not suitable because its values do not lie within the control limits. This conclusion also coincides with a preliminary prediction made on the basis of a visual assessment of the graphs in Fig. 3. $D1$ as well as $D2$ shape distributions can certainly be applied in similarity of triangular meshes measurement because their characteristics are almost constant and lie in the control zone.

Shape distri-bution	Scanner	Measurement					$\bar{R}_i$
		$M1$	$M2$	$M3$	$M4$	$M5$	
$D1$	$S1$	0.0860	0.0863	0.0774	0.0746	0.0795	

	$S2$	0.1531	0.1541	0.1563	0.1520	0.1535	
	$S3$	0.1065	0.1032	0.0997	0.1092	0.1052	
	R_{ij}	0.0670	0.0679	0.0789	0.0774	0.0740	0.0731
$D2$	$S1$	0.0334	0.0338	0.0474	0.0379	0.0385	
	$S2$	0.0701	0.0722	0.0701	0.0669	0.0703	
	$S3$	0.0554	0.0546	0.0522	0.0577	0.0595	
	R_{ij}	0.0367	0.0383	0.0227	0.0290	0.0318	0.0317
$D3$	$S1$	1.1977	0.3037	0.5002	0.4866	0.6195	
	$S2$	0.4455	0.3295	0.3786	0.3554	0.3758	
	$S3$	0.6411	0.6662	0.6783	0.6512	0.6701	
	R_{ij}	0.7522	0.3625	0.2998	0.2958	0.2943	0.4009

Tab. 1. MSA calculations applied on Minkowski L_1 norms for all scanned meshes

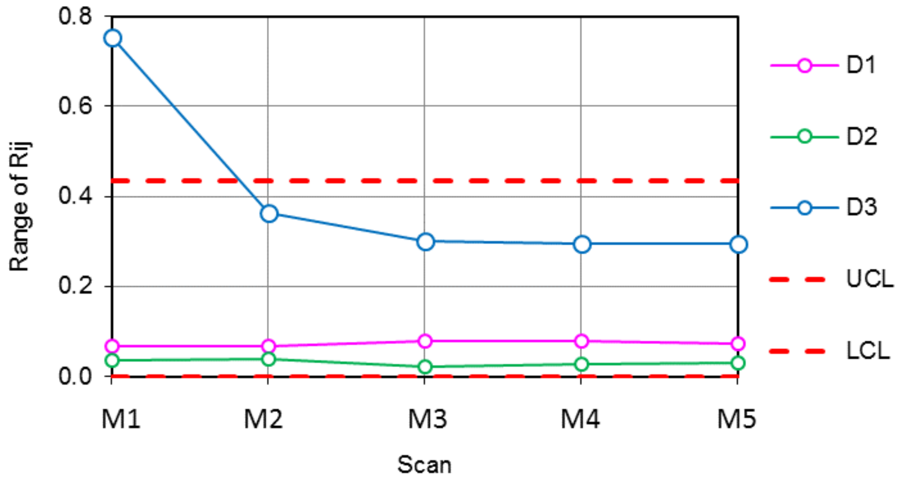

Fig. 4. MSA Range chart

4 Conclusions

In this paper, a measurement system analysis (MSA) was used to investigate the effect of three different shape distributions in the process of triangular meshes similarity measurement. $D1$, $D2$ and $D3$ shape functions were defined and applied on fifteen triangular meshes obtained by repeated optical scanning of the same object by three different scanners to obtain the corresponding shape distributions. To compare the shape distributions of the scanned meshes with a suitable reference data, the shape distribution of the nominal triangular mesh generated on the theoretical CAD model of the scanned object was created. The Minkowski L_1 norm was calculated and all the obtained values were analysed by MSA. Based on MSA results, the D1 and D2 shape distribution were identified as very good tool for similarity of triangular meshes measurement. The $D3$ shape distribution cannot be recommended due to its large variance.

Acknowledgement

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Abstracts

B. Jović, D. Velichová, M. Cvjetić: Golden section: applications in domain of landscape architecture

This paper deals with the analysis of the relationship between natural structures and golden cross sections, and application of the golden cross section in the domain of landscape architecture. The aspects and research results shown in this paper are concerning the geometric construction of the golden section and its applications by the elements of visual aesthetics in landscape architecture as the basic elements of the composition. The aim of the paper is the systematization of elements from the aspect of using the golden section, as well as the application on concrete examples in domain of landscape architecture.

P. Magnaghi-Delfino, T. Norando: Teaching calculus with Maria Gaetana Agnesi

In 2018, we celebrated the three hundredth anniversary of the birth of Maria Gaetana Agnesi, mathematician and benefactress, born in Milan (Italy). We have examined the *Analytical Institutions*, the main work of Maria Gaetana, that she dedicated to students' education. We think that pre-university students can acquire the fundamental mathematical ideas in Differential Calculus using methods and ideas proposed in the books that go back to the origins of the Analysis. From this point of view, we can use many suggestions and examples, contained in Agnesi's Books [1].

C. Păcurar: Reinterpreting Pablo Picasso's paintings using Bézier curves

The present paper is composed of two parts concerning Pablo Picasso's artworks reinterpreted through Bézier curves. For the first part, we aim to reproduce a famous line drawing of Pablo Picasso with the use of Bézier curves. Moreover, we also give another representation for the drawing from a point of view more specific to the Cubism artistic movement. In the second part, we analyze one of the portraits of Dora Maar and try to find another appearance for the painting through an inverse process from the one applied in the first part.

N. Pajerová, I. Linkeová: Shape distribution approach to measure similarity of triangular meshes

The need to measure a similarity between two 3D objects belongs to highly challenging problems in computer graphics. In this paper, a shape distributions approach based on three different shape functions is used to compute a similarity measure of triangular meshes obtained by optical scanning. The evaluation based on a measurement system analysis (MSA), a statistical method commonly used in quality management systems, is applied to assess the capability of individual shape distributions.

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