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G
Slovenský časopis pre geometriu a grafiku

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Slovenská spoločnosť pre Geometriu a Grafiku

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SLOVENSKÁ SPOLOČNOSŤ

SG

PRE GEOMETRIU A GRAFIKU

Nezisková vedecká spoločnosť pre rozvoj geometrie a počítačovej grafiky

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Cieľom spoločnosti je stimulovať vedecký výskum, aplikácie i pedagogickú prácu a metodiku vyučovania v oblasti geometrie a počítačovej grafiky.

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a) podporovať rozvoj geometrie a počítačovej grafiky a ich vzájomnej interakcie

b) presadzovať kvalitu geometrického a grafického vzdelania na všetkých typoch škôl v SR

c) spolupracovať s medzinárodnými spoločnosťami a organizáciami rovnakého zamerania

d) podieľať sa na organizácii vedeckých podujatí, konferencií, seminárov a sympózií o geometrii a počítačovej grafike

e) publikovať vedecký časopis s názvom G venovaný geometrii a grafike

f) rozvíjať vlastnú edičnú a publikačnú činnosť

g) získať priazeň a členstvo organizácií aj jednotlivcov.

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Problémy globálnej variačnej geometrie

Ján Brajerčík

Abstrakt
Globálna variačná geometria je moderná matematická disciplína integrujúca poznatky viacerých oblastí matematiky. Zovšeobecňuje klasický variačný počet, pričom predmetom jej skúmania je geometrická štruktúra pozostávajúca z hladkej variety a diferenciálnej formy na nej definovanej. V tomto príspevku charakterizujeme základné pojmy globálnej variačnej geometrie. Tiež rozoberáme niektoré problémy riešené metódami globálnej variačnej geometrie.

Kľúčové slová: globálna variačná geometria, variačný počet, fibrovaná varieta, jet, Lagrangián, Eulerove-Lagrangeove rovnice

Abstract
We introduce the global variational geometry as a modern mathematical discipline integrating the knowledge of many areas of mathematics. It generalizes classical calculus of variations, and its subject is a geometric structure consisting of a smooth manifold endowed with a differential form. In this contribution, basic concepts of the global variational geometry are characterized. We also discuss some problems solved by methods of the global variational geometry.

Keywords: global variational geometry, calculus of variations, fibered manifold, jet, Lagrangian, Euler-Lagrange equations

1 Úvod

Globálna variačná geometria je odvetvie matematiky, ktoré sa venuje extremálnym problémom, na rozhraní diferenciálnej geometrie, topológie, globálnej analýzy, algebry, variačného počtu a matematickej fyziky. Táto disciplína zovšeobecňuje klasický variačný počet v tom zmysle, že podkladové priestory, ktorými sú Euklidovské priestory, sa nahrádzajú *hladkými* alebo *fibrovanými varietami*, a integrandy variačných funcionálov, ktorými sú Lagrangeove funkcie, sa nahrádzajú *Lagrangeovými diferenciánymi formami*. Predmetom globálnej variačnej geometrie je štúdium extrémov *integrálnych variačných funkcionálov* definovaných pre rezy fibrovaných variet, príslušných diferenciálnych rovníc a objektov invariantných voči transformáciám podkladových geometrických štruktúr.

Základné geometrické idey, ktoré umožnili globalizovať klasický variačný počet, vychádzajú z konceptov E. Cartana [2] pre variačný počet jednorozmerných integrálov a najmä z práce Lepagea [14]. K formovaniu globálnej teórie prispeli hlavne Dedecker [3] (geometrický prístup k variačnému počtu), Garcia [5] (Poincarého-Cartanova forma a invariantné geometrické operácie), Goldschmidt a Sternberg [7] (Cartanova forma, Hamiltonova teória), Krupka [11] (Lepageove formy, variačné funkcionály vyšších rádov) a Trautman [15] (invariancia Lagrangeových systémov). Pre získanie širšieho prehľadu o globálnej variačnej geometrii odporúčame publikáciu Krupka [10].

Štruktúra tohto príspevku je nasledovná. V časti 2 stručne uvedieme základné pojmy klasického variačného počtu, ktoré sú ilustrované niekoľkými príkladmi jednoduchých variačných problémov a ich riešeniami. Časť 3 je venovaná popisu topologických, hladkých a fibrovaných variet. V časti 4 predstavujeme pojem jetu a jeho využitie na konštrukciu tzv. jetových prolongácií fibrovanej variety ako hlavnej podkladovej štruktúry pre globálnu variačnú geometriu. Štruktúra integrálnych variačných funkcionálov je charakterizovaná v časti 5, pričom sú v nej zahrnuté pojmy ako Lagrangián a Eulerove-Lagrangeove rovnice. V časti 6 rozoberáme vybrané problémy globálnej variačnej geometrie.

2 Variačný počet

Kľúčovú úlohu vo variačnom počte zohráva pojem funkcionálu. Pod *funkcionálom* máme na mysli priradenie, pri ktorom každému zobrazeniu z danej triedy zobrazení prislúcha reálne číslo. *Variačný počet* sa potom zaoberá hľadaním maxima a minima funkcionálov.

Teraz uvedieme niekoľko typických príkladov *variačných problémov*, teda problémov, ktoré sa týkajú určenia extremálnych hodnôt funkcionálov.

1. *Nájdi najkratšiu krivku v rovine spájajúcu jej dva body A a B; teda nájdi krivku $y = y(x)$, pre ktorú funkcionál*

$$\int_a^b \sqrt{1+(y')^2}\,\mathrm{d}x$$

dosahuje svoje minimum.

Ukazuje sa, že hľadanou krivkou je úsečka spájajúca body A a B.

2. *Spomedzi všetkých kriviek spájajúcich dva pevné body $A = (x_A, y_A)$ a $B = (x_B, y_B)$ nájdi takú krivku, aby častici pohybujúcej sa po nej pod vplyvom gravitácie trvalo čo najkratší čas dostať sa z A do B.*

Hľadáme krivku $y = y(x)$, pre ktorú časový funkcionál

$$T = \int_A^B \mathrm{d}t = \int_{x_A}^{x_B} \frac{\mathrm{d}s}{v},$$

dosahuje svoje minimum. Keďže

$$\mathrm{d}s = \sqrt{1+(y')^2}\,\mathrm{d}x, \quad v = \sqrt{2gy},$$

potom čas T pohybu častice z A do B je

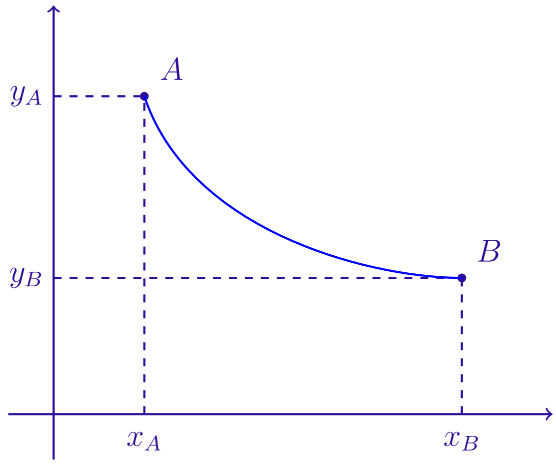
$$T = \int_{x_A}^{x_B} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,\mathrm{d}x.$$

Hľadaná krivka sa nazýva *brachistochrona* (z gréckeho *brachistos* - najkratší, *chronos* - čas).

Ukazuje sa, že ide o časť cykloidy ležiacej vo vertikálnej rovine, ktorá prechádza bodmi A a B. Táto krivka (obr. 1) je parametricky vyjadrená rovnicami

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta),$$

kde hodnota r je určená počiatočnými podmienkami.



Obr. 1. Brachistochrona

Problém brachistochrony bol pôvodne formulovaný Johannom Bernoullim v roku 1696 a zohral dôležitú úlohu v rozvoji variačného počtu. Okrem Johanna Bernoulliho problém vyriešili aj Jacob Bernoulli, Newton a L'Hospital.

3. *Spomedzi všetkých kriviek spájajúcich dané body A a B nájdi krivku, ktorá rotáciou okolo osi x vytvorí plochu s najmenším obsahom.*

Ako je dobre známe, obsah plochy vzniknutej rotáciou krivky $y = y(x)$ okolo osi x je Keďže

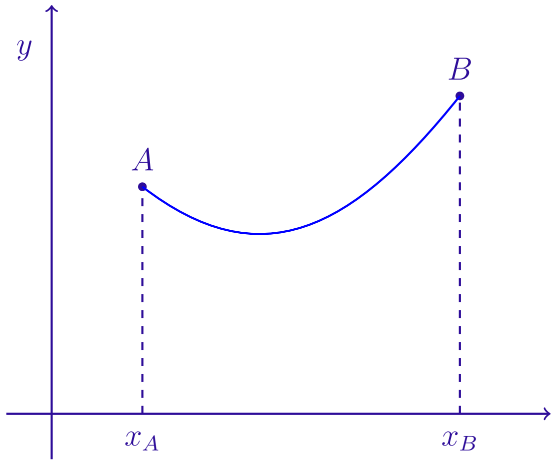
$$2\pi \int_{x_A}^{x_B} y\sqrt{1 + (y')^2}\, \mathrm{d}x.$$

Hľadaná krivka je tzv. *reťazovka* prechádzajúca bodmi A, B. Názov krivky pochádza z tvaru, ktorý by zaujala hmotná reťaz ukotvená v bodoch A, B vo vertikálnej rovine obsahujúcej tieto body pod vplyvom tiaže. Reťazovka je vyjadrená predpisom

$$y = C \cosh \frac{x + C_1}{C},$$

pričom konštanty C, C_1 sú určené počiatočnými podmienkami $y_A = y(x_A)$, $y_B = y(x_B)$ reprezentujúcimi polohu bodov A, B (v prípade špeciálnej polohy bodov A, B riešenie degeneruje; pre detailnejší opis situácie pozri Gelfand a Fomin [6]).

Rotačná plocha vzniknutá rotáciou reťazovky sa nazýva *katenoid*. Tento problém je tiež známy ako *problém minimálnych plôch* a bol vyriešený Leibnizom, Huygensom a Johannom Bernoullim v roku 1691.



Obr. 2. Reťazovka

Všetky vyššie uvedené problémy sa týkajú integrálov, ktoré možno zapísať v tvare

$$\int_a^b L(x, y, y')\mathrm{d}x,$$

kde funkcia $L : V \to \mathbf{R}$, $V \subset \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ sa nazýva *Lagrangeova funkcia* (alebo *Lagrangián*). Lagrangeova funkcia L definuje na množine všetkých prípustných zobrazení $y = \gamma(x)$ *akciu* S *funkcie Lagrangiánu* L na intervale $[a, b]$ danú predpisom

$$S : \gamma \mapsto \int_a^b (L \circ \gamma)(x)\mathrm{d}x \in \mathbf{R}. \tag{1}$$

Zobrazenie γ, pre ktoré funkcionál S nadobúda maximum alebo minimum, sa nazýva *extremála* S. Extremála akcie (1) je riešením *Eulerovej-Lagrangeovej rovnice*

$$\left(\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'}\right) \circ \gamma = 0. \tag{2}$$

Ako príklad uvažujme Lagrangeovu funkciu $L : \mathbf{R}^3 \to \mathbf{R}$ danú predpisom

$$L(t, q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - mgq \tag{3}$$

pre kladné konštanty m, g. Príslušná Eulerova-Lagrangeova rovnica je $\ddot{q} = -g$ a jej riešenie má tvar

$$q(t) = -\frac{1}{2}gt^2 + c_1 t + c_2 \tag{4}$$

kde konštanty c_1, c_2 sú určené počiatočnými podmienkami. Lagrangián L (3) predstavuje rozdiel kinetickej a potenciálnej energie častice s hmotnosťou m v gravitačnom poli s gravitačným zrýchlením g. Riešenie (4) reprezentuje voľný pád častice v gravitačnom poli.

Vo všeobecnosti, ak uvažujeme mechanický systém s kinetickou energiou T a potenciálnou energiou V, potom extremála variačného funkcionálu definovaného Lagrangiánom $L = T - V$ určuje pohyb mechanického systému.

3 Variety

Táto časť je venovaná zavedeniu jednej zo základných podkladových štruktúr globálnej variačnej geometrie, *fibrovanej variete*. Najprv opíšeme pojem *topologickej* a *hladkej variety*.

Nech X je topologický priestor. Hovoríme, že X je topologická varieta dimenzie n (alebo topologická n-varieta), ak má nasledujúce vlastnosti:

- X je *Hausdorffov* priestor, teda každé dva body z X je možné oddeliť otvorenými množinami z X,
- X je *spočítateľný druhého typu*, teda topológia na X má spočítateľnú bázu,
- X je *lokálne Euklidovský* dimenzie n, teda ku každému $x \in X$ existuje okolie U bodu x a homeomorfizmus $\varphi : U \to \widetilde{U}$ z U do $\widetilde{U} = \varphi(U) \subset \mathbf{R}^n$.

Triviálnym príkladom topologickej variety dimenzie n je $\mathbf{R}^n$ so štandardnou topológiou otvorených gúľ. Množina $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ uvažovaná s topológiou indukovanou štandardnou topológiou na $\mathbf{R}^2$ je netriviálnym príkladom topologickej variety dimenzie 1.

Nech X je topologická n-varieta. *Súradnicový systém* na X je dvojica (U, φ), kde U je otvorená podmnožina X a $\varphi : U \to \widetilde{U}$ je homeomorfizmus z U do $\widetilde{U} = \varphi(U) \subset \mathbf{R}^n$. Priamo z definície topologickej variety máme, že ku každému jej bodu existuje nejaký súradnicový systém obsahujúci tento bod. Topologickú varietu je teda možné pokryť súradnicovými systémami, ktoré nám nejakým spôsobom varietu popisujú. Napríklad, ako súradnicový systém pre ľubovoľný bod $x \in \mathbf{R}^n$ možno vziať *globálny* súradnicový systém $(\mathbf{R}^n, \mathrm{id}_{\mathbf{R}^n})$. Vo všeobecnosti, na topologickej variete nemusí existovať globálny súradnicový systém.

Na zabezpečenie prechodu z jedného súradnicového systému do iného definujeme pojem hladkosti funkcie. Nech U a V sú otvorené podmnožiny Euklidovského priestoru $\mathbf{R}^n$, resp. $\mathbf{R}^m$. Funkcia $F : U \to V$ sa nazýva *hladká*, ak každá jej zložka má spojité parciálne derivácie všetkých rádov. Ak F je navyše bijektívna a má hladké inverzné zobrazenie, nazýva sa *difeomorfizmus*. Zrejme, ak F je difeomorfizmus, potom $n = m$.

Nech X je topologická n-varieta. Dva súradnicové systémy (U, φ), (V, ψ) sa nazývajú *hladko kompatibilné*, ak buď $U \cap V = \emptyset$, alebo kompozícia $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ je difeomorfizmus.

Analogicky snahe popísať povrch Zeme pomocou jednotlivých máp definujeme pojem atlasu na variete. Hladký atlas na topologickej variete X je súbor hladko kompatibilných súradnicových systémov, ktorých definičné obory pokrývajú X. K danému atlasu na X môžeme pridávať ďalšie hladko kompatibilné súradnicové systémy až kým nezískame maximálny hladký atlas na X. *Hladká* (alebo *diferencovateľná* alebo C^∞-) *štruktúra* na topologickej n-variete je potom definovaná ako maximálny hladký atlas na X. *Hladká varieta* je dvojica $(X, \mathcal{A})$, kde X je topologická varieta a $\mathcal{A}$ je hladká štruktúra na X.

Vo všeobecnosti môžeme na tolopogickej variete X vybrať viac hladkých atlasov reprezentujúcich rovnakú hladkú štruktúru. Napríklad, na variete S^1 máme hladký atlas pozostávajúci z dvoch súradnicových systémov určených stereografickými projekciami na príslušné množiny, alebo môžeme vziať hladký atlas pozostávajúci zo štyroch súradnicových systémov, množinami ktorých sú otvorené polkružnice, a projekcie týchto množín na príslušné súradnicové osi v $\mathbf{R}^2$ predstavujú homeomorfizmy na $(-1, 1) \subset \mathbf{R}$.

Existujú príklady topologických priestorov umožňujúcich definovať viac nekompatibilných hladkých štruktúr; medzi ne patrí napríklad aj $\mathbf{R}$ so štandardnou topológiou. Na druhej strane, existujú topologické priestory, na ktorých neexistuje hladká štruktúra (pozri napríklad [13]).

Teraz rozšírime definíciu hladkého zobrazenia medzi otvorenými podmnožinami Euklidovských priestorov na definíciu *hladkej funkcie* na hladkej variete a *hladkého zobrazenia* medzi dvoma hladkými varietami. Ak X je hladká varieta, funkcia $f : X \to \mathbf{R}$ sa nazýva *hladká* ak pre každé $x \in X$ existuje taký súradnicový systém (U, φ) na X, že $x \in U$ a kompozícia $f \circ \varphi^{-1}$ je hladké zobrazenie Euklidovských priestorov. Nech X, Y sú hladké variety a nech $F : X \to Y$ je zobrazenie. Hovoríme, že F je *hladké zobrazenie* ak pre každé $x \in X$ existuje taký súradnicový systém (U, φ) obsahujúci x a taký súradnicový systém (V, ψ) obsahujúci $F(x)$, že $F(U) \subset V$ a kompozícia $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ je hladké zobrazenie Euklidovských priestorov.

Pod pojmom *štruktúra fibrovanej variety* rozumieme hladkú varietu Y, hladkú varietu X a hladkú surjektívnu submerziu $\pi : Y \to X$. Varieta Y so štruktúrou fibrovanej variety sa nazýva (*hladká*) *fibrovaná varieta*. X sa nazýva *báza* a π sa nazýva *projekcia* fibrovanej variety Y. Množina $\pi^{-1}(x)$ sa nazýva *fiber* nad x.

Triviálnym príkladom fibrovanej variety je $\pi_1 : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$. Všeobecnejšie, každá varieta v tvare kartézskeho súčinu bázovej variety X a ďalšej hladkej variety je fibrovaná varieta. Na druhej strane, netriviálnymi príkladmi fibrovaných variet sú $\pi_2 : M \to S^1$, kde M je *Möbiov pásik* (obr. 3), alebo $\pi_3 : S^3 \to S^2$, tiež nazývaná *Hopfova fibrácia*, kde S^n označuje n-rozmernú sféru (zovšeobecnenie S^1; pozri časť 3). Pre vizualizáciu Hopfovej fibrácie pozri, napríklad, Zamboj [16].

Obr. 3. Möbiov pásik (zdroj: internet)

Nech $\pi : Y \to X$ je fibrovaná varieta, nech $\dim X = n$, $\dim Y = n + m$. Priamo z definície fibrovanej variety máme, že ku každému bodu $y \in Y$ existuje súradnicový systém (V,ψ), $\psi = (u^i, y^\sigma)$, v bode y, kde $1 \leq i \leq n$, $1 \leq \sigma \leq m$, s nasledovnými vlastnosťami:

(a) Existuje súradnicový systém (U, φ), $\varphi = (x^i)$, v bode $x = \pi(y)$, kde $1 \leq i \leq n$, pomocou ktorého je projekcia π vyjadrená rovnicami $x^i \circ \pi = u^i$.

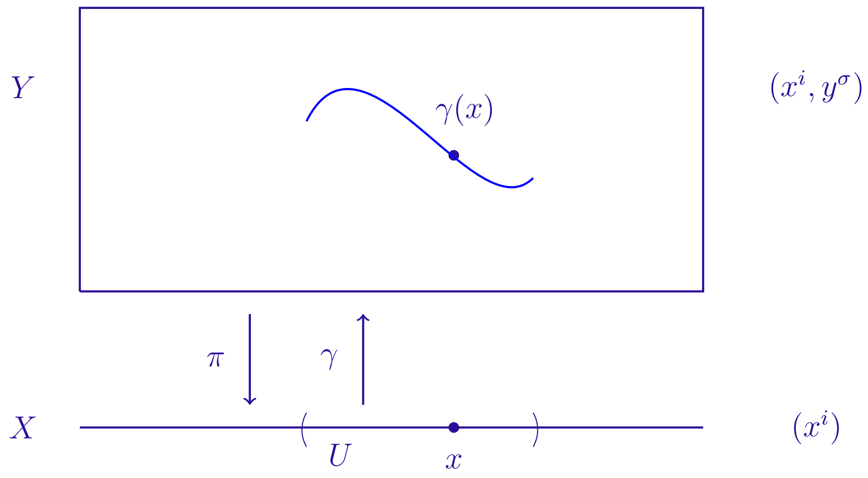
(b) $U = \pi(V)$.

Súradnicový systém (V, ψ) s takýmito vlastnosťami sa nazýva *fibrovaný súradnicový systém* na Y. Súradnicový systém (U, φ) je potom jednoznačne definovaný a nazýva sa *asociovaný* so súradnicovým systémom (V, ψ). Zvyčajne píšeme (V, ψ), $\psi = (x^i, y^\sigma)$, teda píšeme x^i miesto u^i.

Rez fibrovanej variety $\pi : Y \to X$ je také zobrazenie $\gamma : U \to Y$, kde $U \subset X$ je otvorená množina, že $\pi \circ \gamma = \mathrm{id}_U$. Vo fibrovanom súradnicovom systéme (V, ψ), $\psi = (x^i, y^\sigma)$, na Y,

$$x^i \circ \gamma = x^i, \quad y^\sigma \circ \gamma = f^\sigma(x^i).$$

Na ilustráciu fibrovanej variety sa často používa príklad fibrovanej variety $\pi_1 : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ pre $n = 1$ (obr. 4).



Obr. 4. Fibrovaná varieta a jej rez

4 Jety

Cieľom v tejto časti je zaviesť pojem *jetového predĺženia fibrovanej variety*. Pre tento účel najprv opíšeme pojem *jetu zobrazenia*.

Uvažujme najskôr nasledujúci príklad. Nech $f : \mathbf{R} \to \mathbf{R}$ je zobrazenie a nech $x_0 \in \mathcal{D}(f)$. 2-*jet zobrazenia* f *v bode* x_0 je usporiadaná štvorica reálnych čísel

$$(x_0, f(x_0), Df(x_0), D^2f(x_0))$$

a označuje sa $J^2_{x_0}f$. Pre prípad $f(x) = \sin x$ a $x_0 = \frac{\pi}{2}$ je $J^2_{x_0}f = (\frac{\pi}{2}, 1, 0, -1)$. Môžeme nájsť aj iné zobrazenie $g : \mathbf{R} \to \mathbf{R}$ tak, že $J^2_{x_0}f = J^2_{x_0}g$. Stačí vziať časť Taylorovho radu funkcie sin s príslušnými koeficientami, napríklad $g(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$.

Nech $\pi : Y \to X$ je fibrovaná varieta, nech $y \in Y$, $x = \pi(y)$. Nech r je kladné celé číslo. Označme $\Gamma^r_{x,y}$ množinu všetkých takých C^r-rezov γ variety Y definovaných v bode x, že $\gamma(x) = y$. Uvažujme binárnu reláciu $\sim$ na $\Gamma^r_{x,y}$: „$\gamma_1 \sim \gamma_2$ práve vtedy, ak existuje taký súradnicový systém (V, ψ), $\psi = (x^i, y^\sigma)$, na Y, že

$$D_{i_1}D_{i_2}\dots D_{i_k}(y^\sigma\gamma_1\varphi^{-1})(\varphi(x)) = D_{i_1}D_{i_2}\dots D_{i_k}(y^\sigma\gamma_2\varphi^{-1})(\varphi(x))$$

pre všetky $k = 1, 2, \dots, r$, a všetky $1 \le i_1 \le i_2 \le \dots \le i_k \le n$“.

Relácia $\sim$ je reláciou ekvivalencie na $\Gamma^r_{x,y}$, ktorá nezávisí od voľby fibrovaného súradnicového systému. Trieda ekvivalencie, ktorej reprezentantom je $\gamma \in \Gamma^r_{x,y}$, sa nazýva r-*jet* zobrazenia γ s *počiatkom* v bode x a *koncom* v bode y; označuje sa $J^r_x\gamma$.

J^rY označuje množinu všetkých r-jetov s počiatkom v X a koncom v Y. Štruktúra fibrovanej variety Y indukuje hladkú štruktúru na J^rY a $\pi^r : J^rY \to X$ s touto štruktúrou je fibrovaná varieta. J^rY sa potom nazýva *r-té jetové predĺženie* variety Y.

Ak γ je hladký rez variety Y, potom hladké zobrazenie

$$J^r\gamma : x \mapsto J^r\gamma(x) = J^r_x\gamma$$

sa nazýva *r-té jetové predĺženie* zobrazenia γ (ide o rez fibrovanej variety $\pi^r : J^rY \to X$).

5 Variačné štruktúry na fibrovanej variete

Objekt, ktorý nahrádza Lagrangeovu funkciu vo variačnom funkcionáli (1) je *diferenciálna forma*. Nech X je n-rozmerná hladká varieta a nech T_xX označuje *dotykový priestor* variety X v bode $x \in X$. Na vektorovom priestore T_xX uvažujme vektorový priestor Λ^kT_xX k-foriem, teda tenzorov typu $(0, k)$. Priestor

$$\Lambda^kTX = \bigcup_{x\in X} \Lambda^kT_xX$$

nazývame *priestorom k-foriem* na X. *Diferenciálna k-forma* na $U \subset X$ je potom hladké zobrazenie

$$\eta : U \to \Lambda^kTX.$$

Lagrangián (rádu r) pre fibrovanú varietu $\pi : Y \to X$, $\dim X = n$, je *π^r-horizontálna* n-forma definovaná na otvorenej podmnožine J^rY. Požiadavka horizontálnosti vzhľadom k projekcii $\pi^r : J^rY \to X$ znamená, že vo fibrovanom súradnicovom systéme (V, ψ), $\psi = (x^i, y^\sigma)$, na Y, je Lagrangián λ vyjadrený v tvare

$$\lambda = \mathcal{L}\,\omega_0, \quad \omega_0 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n,$$

kde $\mathcal{L} : J^rY \to \mathbb{R}$ je *Lagrangeova funkcia* asociovaná so súradnicovým systémom (V, ψ). *Variačná štruktúra* je potom dvojica (Y, λ), kde Y je fibrovaná varieta nad n-rozmernou bázou X s projekciou π a λ je Lagrangián pre Y.

Uvažujme variačnú štruktúru (Y, λ). Nech Ω je n-rozmerná kompaktná podvarieta variety X s hranicou (takéto Ω nazývame tiež *kúsok* variety X). Označme $\Gamma_\Omega(\pi)$ množinu hladkých rezov variety Y definovaných nad kúskom Ω. Dostávame funkciu $\Gamma_\Omega(\pi) \ni \gamma \mapsto \lambda_\Omega(\gamma) \in \mathbf{R}$ definovanú predpisom

$$\lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda,$$

kde *pull-back* $J^r\gamma^*$ zobrazenia $J^r\gamma$ zobrazí Lagrangián λ na n-formu na báze X. Funkcia λ_Ω

sa nazýva *variačný funkcionál*, asociovaný s variačnou štruktúrou (Y, λ) (nad Ω).

Pomocou geometrických operátorov akými sú vonkajšia derivácia diferenciálnej formy a rozklad diferenciálnej formy na kontaktné komponenty priraďujeme Lagrangiánu λ jeho *Eulerovu-Lagrangeovu formu* E_λ. Zobrazenie $\lambda \to E_\lambda$ sa nazýva *Eulerovo-Lagrangeovo zobrazenie*. Ak λ je n-forma rádu r, potom E_λ je $(n+1)$-forma rádu $2r$. Ak zvolíme fibrovaný súradnicový systém (V, ψ), $\psi = (x^i, y^\sigma)$ na variete Y, potom komponenty $E_\sigma(\mathcal{L})$ Eulerovej-Lagrangeovej formy E_λ v asociovanom súradnicovom systéme na J^rY sa nazývajú *Eulerove-Lagrangeove výrazy*.

Jedna z úloh globálnej variačnej geometrie je štúdium extremálnych hodnôt, *extremál*, daného variačného funkcionálu. Extremály hľadáme s využitím nasledovného tvrdenia.

Veta 5.1. Nech λ je Lagrangián r-tého rádu pre fibrovanú varietu Y. Nech $\gamma : U \to Y$ je rez Y. Nasledujúce podmienky sú ekvivalentné.

(a) γ je extremála variačného funkcionálu λ_Ω.

(b) Pre každý taký fibrovaný súradnicový systém (V, ψ), $\psi = (x^i, y^\sigma)$, že $\pi(V) = U$, γ spĺňa systém parciálnych diferenciálnych rovníc v tvare

$$E_\sigma(\mathcal{L}) \circ J^{2r}\gamma = 0, \quad 1 \leq \sigma \leq m. \tag{5}$$

Rovnice (5) sa nazývajú *Eulerove-Lagrangeove rovnice*. Podrobnejší prehľadu o variačných štruktúrach na fibrovaných varietách možno nájsť v [10].

Ak, napríklad, uvažujeme variačný problém definovaný Lagrangiánom prvého rádu

$$\lambda = \mathcal{L}\omega_0, \quad \mathcal{L} = \mathcal{L}(x^i, y^\sigma, y^\sigma_j),$$

Eulerove-Lagrangeove rovnice sú v tvare

$$\left(\frac{\partial \mathcal{L}}{\partial y^\sigma} - \frac{\partial^2 \mathcal{L}}{\partial x^i \partial y^\sigma_i} - \frac{\partial^2 \mathcal{L}}{\partial y^\nu \partial y^\sigma_i} y^\nu_i - \frac{\partial^2 \mathcal{L}}{\partial y^\nu_j \partial y^\sigma_i} y^\nu_{ji}\right) \circ J^2\gamma = 0, \tag{6}$$

kde $1 \leq \sigma \leq m$, pričom m je dimenzia každého fibra variety Y (v tomto aj nasledujúcich vzorcoch uplatňujeme Einsteinovu sumačnú konvenciu). Rovnice (6) sa pre $n = m = 1$ redukujú na jedinú rovnicu, ktorá korešponduje s rovnicou (2).

6 Vybrané problémy

V tejto časti uvedieme problémy, v riešení ktorých sú použité pojmy a metódy globálnej variačnej geometrie.

Riešenia Einsteinových rovníc vo všeobecnej teórii relativity

Prvý z nich sa týka charakteristiky extremál *Hilbertovho variačného funkcionálu*, ktorý bol formulovaný D. Hilbertom (1915) v [8]. Ide o variačný funkcionál pre metrické polia na ľubovoľnej n-rozmernej variete X. Vo všeobecnej teórii relativity sa pre $n = 4$ príslušné Eulerove-Lagrangeove rovnice nazývajú *Einsteinovými rovnicami vo vákuu*.

Nech X je n-rozmerná hladká varieta, nech T_2^0X je priestor tenzorov typu $(0,2)$ nad X. Uvažujme $\mathrm{Met}\,X$ ako otvorenú podmnožinu T_2^0X *symmetrických, regulárnych bilineárnych foriem* na T_xX, kde $x \in X$. Rezy fibrovanej variety $\tau : \mathrm{Met}\,X \to X$ sú *metrické polia* na X. Integrálne variačné funkcionály pre metrické polia sú definované n-formami na $J^r\mathrm{Met}\,X$.

Súradnicový systém (U,φ), $\varphi = (x^i)$, na X, indukuje súradnicový systém (V,ψ), $\psi = (x^i, g_{ij})$, na $\mathrm{Met}\,X$, kde $V = \tau^{-1}(U)$ a g_{ij} sú funkcie na V definované vzťahom

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad g_{ij} = g_{ji}, \quad \det(g_{ij}) \neq 0.$$

Funkcie

$$\Gamma^k_{ij} = \tfrac{1}{2}\, g^{kl}\left(\tfrac{\partial g_{il}}{\partial x^j} + \tfrac{\partial g_{jl}}{\partial x^i} - \tfrac{\partial g_{ij}}{\partial x^l}\right),$$

kde g^{kl} sú prvky inverznej matice k matici g_{ij}, sa nazývajú *Christoffelove symboly*. Výrazy

$$R_{ik} = \tfrac{\partial \Gamma^l_{ik}}{\partial x^l} - \tfrac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^l_{km}\Gamma^m_{il}, \qquad R = g^{ik}R_{ik},$$

definujú *Ricciho tenzor* s komponentmi R_{ik}, resp. funkciu $R : J^2\mathrm{Met}\,X \to \mathbf{R}$, nazývanú *skalárna krivosť*. *Hilbertov Lagrangián* je daný vzťahom

$$\lambda = R\sqrt{|\det(g_{ij})|} \cdot \omega_0$$

a príslušný variačný funkcionál

$$\Gamma_\Omega(\tau) \ni g \mapsto \lambda_\Omega(g) = \textstyle\int_\Omega J^2g^*\lambda \in \mathbf{R}$$

sa nazýva *Hilbertov variačný funkcionál* pre metrické polia na X. Príslušné Eulerove-Lagrangeove rovnice (Einsteinove rovnice vo vákuu) sú určené formulou

$$\left(R_{ij} - \frac{1}{2}R\,g_{ij}\right) \circ J^2g = 0. \tag{7}$$

Známe riešenie rovníc (7) je *Schwarzschildova metrika* (1916), v sférických súradniciach (t,r,φ,ϑ) na príslušnej podmnožine $\mathbf{R} \times \mathbf{R}^3 \backslash \{(0,0,0)\}$ určená ako

$$g = -\left(1 - \frac{C_0}{r}\right)\mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{C_0}{r}\right)^{-1}\mathrm{d}r \otimes \mathrm{d}r + r^2(\sin^2\vartheta\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi + \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta), \tag{8}$$

kde konštanta C_0 je určená fyzikálnymi vlastnosťami hmotného telesa, v okolí ktorého je definované riešenie.

Úplnou charakteristikou riešení rovníc (7) sa v práci [12] zaoberali Krupka a Brajerčík. Získaná množina riešení je parametrizovaná rýdzomonotónnymi funkciami $q = q(r)$, kde r je radiálna súradnica, a reálnymi parametrami C a C', ktoré majú význam integračných konštánt. V súradniciach (t,q,φ,ϑ), analogických sférickým súradniciam (t,r,φ,ϑ) na $\mathbf{R} \times \mathbf{R}^3 \backslash \{(0,0,0)\}$, sú riešenia g rovníc (7) zapísané v tvare

$$g = C' \left(1 - \tfrac{C}{q}\right) \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \tfrac{C}{q}\right)^{-1} \mathrm{d}q \otimes \mathrm{d}q + q^2(\sin^2 \vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi + \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta).$$

Množina riešení je odvodená bez požiadavky na signatúru hľadanej metriky g, čím sa líši od štandardného prístupu pri odvodení Schwarzschildovej metriky, keď sa predpokladá signatúra metriky Lorentzovského typu.

Pre špeciálnu voľbu $q = r,\ \ C = C_0,\ \ C' = -1$ dostávame Schwarzschildovu metriku (8).

Hľadanie riešení Einsteinových rovníc nezávislých na voľbe bázovej variety X daného variačného problému a hľadanie riešení zovšeobecnených rovníc (7) patrí medzi otvorené otázky všeobecnej teórie relativity.

Inverzný variačný problém

Ďalší známy problém globálnej variačnej geometrie je *inverzný problém variačného počtu*. Zhruba povedané, tento problém spočíva v nájdení podmienok zaručujúcich existenciu Lagrangiánu, ktorého Eulerove-Lagrangeove rovnice incidujú s daným systémom diferenciálnych rovníc. Ak sú tieto podmienky splnené, následnou úlohou je nájsť *všetky* Lagrangiány pre daný systém rovníc.

Uvažujme systém m obyčajných diferenciálnych rovníc druhého rádu v implicitnej forme

$$\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j) = 0, \quad i, j = 1, 2, \dots, m,$$

pre m funkcií reálnej premennej $t \mapsto x^j(t)$. Hovoríme, že systém funkcií $\varepsilon = \{\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)\}$ je *variačný*, ak existuje funkcia $\mathcal{L} = \mathcal{L}(x^j, \dot{x}^j)$ tak, že

$$\varepsilon_i = \tfrac{\partial \mathcal{L}}{\partial x^i} - \tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\partial \mathcal{L}}{\partial \dot{x}^i}, \quad i = 1, 2, \dots, m.$$

Veta 6.1. Nasledujúce dve podmienky sú ekvivalentné.

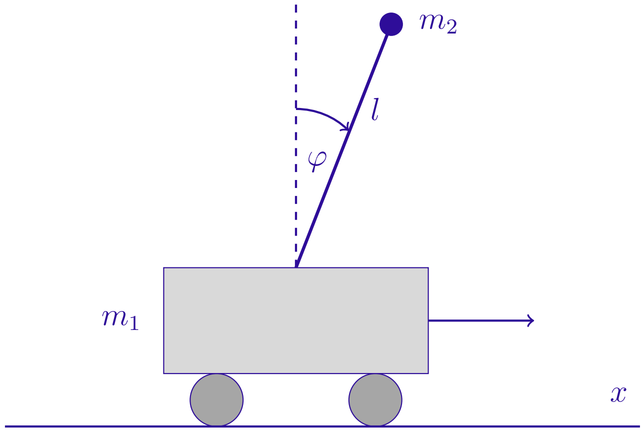
(a) Systém $\varepsilon = \{\varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)\}$ je variačný.

(b) Funkcie ε_i spĺňajú rovnice

$$\begin{aligned}
&\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} - \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} = 0, \\
&\frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i}\right) = 0, \\
&\frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i}\right) = 0.
\end{aligned} \tag{9}$$

Systém (9) sa nazýva *Helmholtzove podmienky* (Helmholtz, 1887). Ak systém funkcií ε spĺňa tieto podmienky, pre hľadaný Lagrangián môžeme vziať *Vainbergov-Tontiho Lagrangián* (pozri, napríklad, [17]).

Táto teória je aplikovateľná na mnohé variačné problémy. Ako príklad môžeme uviesť známy mechanický systém, inverzné kyvadlo na vozíku (obr. 5).



Obr. 5. Inverzné kyvadlo na vozíku

Systém pozostáva z kyvadla, zloženého z nehmotnej tyče s dĺžkou l, na konci ktorej je hmotný bod s hmotnosťou m_2, upevneného na vozíku s hmotnosťou m_1. Konfiguračná varieta systému je $Q = \mathbf{R} \times S^1$, pričom súradnica x označuje pozíciu vozíka a φ označuje veľkosť uhla medzi kyvadlom a vertikálou.

Lagrangeova funkcia pre systém pohybujúci sa po horizontálnej osi x je

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 l \cos\varphi \dot{x}\dot{\varphi} + \tfrac{1}{2} m_2 l^2 \dot{\varphi}^2 - m_2 g l \cos\varphi.$$

Eulerove-Lagrangeove rovnice pre voľný pohyb systému sú dané formulami

$$\begin{aligned} m_2 l \dot{\varphi}^2 \sin\varphi - (m_1 + m_2)\ddot{x} - m_2 l \ddot{\varphi} \cos\varphi &= 0, \\ m_2 l (g \sin\varphi - \ddot{x} \cos\varphi - l\ddot{\varphi}) &= 0. \end{aligned}$$

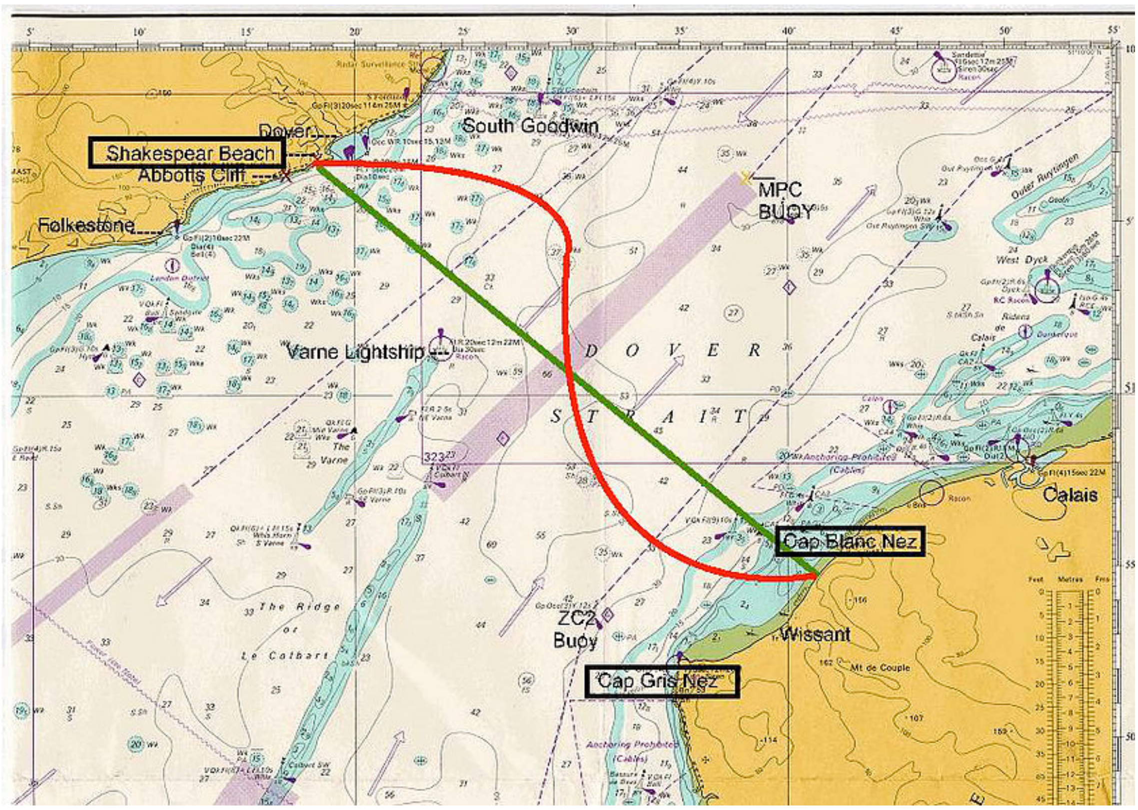
Bez vonkajšej intervencie zrejme kyvadlo padá dole. Naším cieľom je riadiť vozík tak, aby sme stabilizovali kyvadlo v okolí vertikálnej pozície pomocou *variačných síl*. Vo všeobecnosti hľadáme funkcie ϕ_1, ϕ_2 závisiace na $x, \varphi, \dot{x}, \dot{\varphi}$ tak, aby systém rovníc

$$\begin{aligned} m_2 l \dot{\varphi}^2 \sin\varphi - (m_1 + m_2)\ddot{x} - m_2 l \ddot{\varphi} \cos\varphi &= \phi_1, \\ m_2 l (g \sin\varphi - \ddot{x} \cos\varphi - l\ddot{\varphi}) &= \phi_2 \end{aligned}$$

bol variačný a zároveň stabilizoval kyvadlo v okolí vertikálnej pozície. Niektoré výsledky riešenia tohto problému možno nájsť, napríklad v [1] a [4]. Štúdium prípustných variačných síl stabilizujúcich tento a jemu podobné mechanické systémy je objektom aktuálneho výskumu.

Zermelov navigačný problém

Tento klasický problém formuloval E. Zermelo (1931) vo svojej práci [18]. Zermelov navigačný problém patrí medzi najviac skúmané problémy optimálneho riadenia v matematike. Zaoberá sa



Obr. 6. Zermelov navigačný problém (zdroj: internet)

navigáciou loďky pohybujúcej sa na vodnej ploche pod vyplyvom (časovo premenných) vodných prúdov a vetra, ktorú je potrebné dostať z východiskového bodu do cieľa za čo najkratší čas. Je zrejmé, že bez vonkajších vplyvov je trajektóriou zaručujúcou dosiahnutie minimálneho času úsečka spájajúca počiatočný a cieľový bod. Pri uvažovaní vonkajších vplyvov je vo všeobecnosti optimálna trajektória odlišná od úsečky (napríklad ako červená krivka na obr. 6).

Problém bol pre prípady $\mathbf{R}^2$ a $\mathbf{R}^3$ vyriešený samotným Zermelom. Ďalšie výsledky boli získané pre špeciálne prípady, napríklad pre konštantný vplyv vetra. Optimálnu trajektóriu pohybu na všeobecnejších hladkých plochách získal P. Kopacz [9]. V súčasnosti sú formulované všeobecnejšie verzie tohto problému a snahou je hľadať metódy ich riešenia.

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The Equivalence of Side-Angle-Side and Side-Side-Angle in the Absolute Plane

John Donnelly

Abstrakt
Tvrdenie o zhodnosti trojuholníkov, Veta ssu (strana-strana-uhol), nie je vo všeobecnosti kritériom pre dôkaz zhodnosti trojuholníkov. Pomocou kritéria o zhodnosti Veta sus (strana-uhol-strana) však možno dokázať niekoľko tvrdení v geometrii trojuholníka vrátane Vety ssu. V tomto článku ukážeme, že aj opak je pravdivý. Konkrétne, ak predpokladáme platnosť tvrdení, ktoré zahŕňajú Vetu ssu, potom možno dokázať kritérium Veta sus ako ich dôsledok. V dôkazoch sa neuvádzajú žiadne predpoklady o eliptických alebo hyperbolických vlastnostiach rovnobežnosti.

Kľúčové slová: ssu (strana-strana-uhol), trojuholník, súvislá absolútna rovina

Abstract
In general, Side-Side-Angle is not a criterion for congruence of triangles. However, one can use Side-Angle-Side to prove several statements in triangle geometry which involve Side-Side-Angle. We prove that the converse holds as well. Namely, we show that if we assume the statements which involve Side-Side-Angle, then we can prove Side-Angle-Side as a consequence of these assumptions. In these proofs, no assumptions about euclidean or hyperbolic parallel properties are made.

Keywords: Side-Side-Angle, Triangle, Continuous Absolute Plane

1 Introduction

In plane geometry, there are various criteria that one can use to show that two triangles are congruent. Specifically, one can use Side-Angle-Side, Angle-Side-Angle, Side-Angle-Angle, and Side-Side-Side to show congruence of triangles. In addition, in hyperbolic geometry one can also use the Angle-Angle-Angle criterion to show that two triangles are congruent.

One can also use Hypotenuse-Leg to show that two right triangles are congruent. This is a special case of Side-Side-Angle. In general, Side-Side-Angle is not a criterion for congruence of triangles. It is easy to construct an example of a pair of triangles that satisfy the hypotheses of Side-Side-Angle, but which are not congruent to each other. However, if we add the extra hypothesis that the angles which are congruent to each other are not opposite the shortest sides in the respective triangles, then Side-Side-Angle can be used to show that the triangles are congruent. Although Side-Side-Angle is not a criterion for congruence of triangles, it has attracted the interest of various mathematicians [3].

When working in plane geometry, one can choose to use either the euclidean parallel postulate or the hyperbolic parallel postulate. *Absolute geometry* is plane geometry in which we assume no parallel postulate. Absolute geometry can be thought of as being a common ground between Euclidean Geometry and Hyperbolic Geometry, and the properties of Absolute Geometry are satisfied by both Euclidean and Hyperbolic Geometry [8], [13]. Absolute geometry is also referred to as *neutral geometry* [8], [9], [16]. Once we assume axioms of continuity, then we can define the notions of distance and angle measure in the plane (see pages 122-124 in [8]). The result is called a *continuous absolute plane*.

One way to define a continuous absolute plane is to use the synthetic approach given by David Hilbert in *Grundlagen der Geometrie* [10]. In this approach, Hilbert refrains from introducing distance and angle measure until the very end through the use of axioms of continuity [8], [9]. This approach is also used by M. J. Greenberg in *Euclidean and Non-Euclidean Geometries: Development and History*, and by R. Hartshorne in *Geometry: Euclid and Beyond* [8], [9].

Another way to define a continuous absolute plane is to use the metric approach given by G.D. Birkhoff in *A Set of Postulates for Plane Geometry Based on Scale and Protractor*. The approach given by Birkhoff incorporates the notions of distance and angle measure from the start [1]. This approach is also used by G. E. Martin in *The Foundations of Geometry and the Non-Euclidean Plane*, by R. S. Millman and G. D. Parker in *Geometry: A Metric Approach with Models*, by E. E. Moise in *Elementary Geometry from an Advanced Standpoint*, and by G. A. Venema in *Foundations of Geometry* [13], [14], [15], [16].

It can be shown that the continuous absolute plane that one gets from the approach of Hilbert is the same as the continuous absolute plane that one gets from the approach of Birkhoff [2], [8], [13]. In both of these approaches, the Side-Angle-Side criterion for congruence of triangles is assumed. After assuming Side-Angle-Side, then one can prove the Angle-Side-Angle, Side-Angle-Angle, Side-Side-Side, and Hypotenuse-Leg criteria for congruence of triangles as theorems [8], [11], [13], [15].

It is well known that in absolute geometry, the Side-Angle-Side and Angle-Side-Angle criteria are equivalent. In particular, one can assume Angle-Side-Angle and prove that Side-Angle-Side still holds [13].

In [4],[5], and [6], it is shown that if we use the approach of Birkhoff to define a continuous absolute plane, then the Side-Angle-Side and Side-Angle-Angle criteria are equivalent, and the Side-Angle-Side and Side-Side-Side criteria are equivalent. More specifically, if we assume exactly one of either Side-Angle-Angle or Side-Side-Side, then we can prove that Side-Angle-Side still holds.

In [7], a model is constructed showing that if we use the approach of Hilbert to define a non-continuous absolute plane (using the axioms given in [9]), and replace Side-Angle-Side with Angle-Angle-Angle, then in general we cannot prove Side-Angle-Side as a theorem.

In this paper, we show that if we use the approach of Birkhoff to define a continuous absolute plane, then Side-Angle-Side and Side-Side-Angle are equivalent. In particular, if we assume Side-Side-Angle instead of Side-Angle-Side, then we can prove that Side-Angle-Side still holds.

2 Initial Assumptions and Basic Definitions

In this section we state our initial assumptions as well as several definitions that we use throughout the paper.

The Incidence Axioms:
(1) There exist sets $\mathcal{P}$ and $\mathcal{L}$ such that each element of $\mathcal{L}$ is a subset of $\mathcal{P}$.
(2) If P and Q are distinct elements of $\mathcal{P}$, then there exists a unique element of $\mathcal{L}$ which contains both P and Q.
(3) There exist three elements P, Q, and R in $\mathcal{P}$ such that no element of $\mathcal{L}$ contains all three of P, Q, and R.

An element of $\mathcal{P}$ is called a *point*, and an element of $\mathcal{L}$ is called a *line*. Given $P \in \mathcal{P}$ and $l \in \mathcal{L}$, then we say that P is *on* l and that l *passes through* P if $P \in l$. Given two distinct points P and Q, we denote the unique line containing P and Q by $\overleftrightarrow{PQ}$.

The Ruler Postulate:
There exists a function $d : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ such that for each line $l \in \mathcal{L}$, there exists a bijection $f : l \to \mathbb{R}$ with the property that for all $P, Q \in l$, $d(P, Q) = |f(P) - f(Q)|$.

For all $P, Q \in \mathcal{P}$, we denote $d(P, Q)$ by PQ, and we call PQ the *distance* from P to Q. If for a line l, a bijection $f : l \to \mathbb{R}$ is such that for all $P, Q \in l$, $d(P, Q) = |f(P) - f(Q)|$, then f is called a *coordinate system* for l.

We say that a point B is *between* points A and C, and we write $A - B - C$ if: *(1)* A, B, and C are three distinct points, *(2)* A, B, and C are collinear, and *(3)* $AB + BC = AC$. If P and Q are two distinct points, then $\overline{PQ} = \{D \in \overleftrightarrow{PQ} \mid D = P, D = Q, \text{ or } P - D - Q\}$ and $\overrightarrow{PQ} = \{D \in \overleftrightarrow{PQ} \mid D = P, D = Q, P - D - Q, \text{ or } P - Q - D\}$. We call $\overline{PQ}$ the *segment* with endpoints P and Q, and we call $\overrightarrow{PQ}$ a *ray* with vertex P. Two segments $\overline{PQ}$ and $\overline{WZ}$ are said to be *congruent* if $PQ = WZ$, in which case we write $\overline{PQ} \cong \overline{WZ}$. We define the *interior* of $\overline{PQ}$ to be the set of all points D such that $P - D - Q$. We denote the interior of $\overline{PQ}$ by $int(\overline{PQ})$.

The Plane Separation Postulate:
For each $l \in \mathcal{L}$, there exist convex sets $\mathcal{H}_1$ and $\mathcal{H}_2$ such that
(1) $\mathcal{P} \backslash l = \mathcal{H}_1 \cup \mathcal{H}_2$
(2) If $P \in \mathcal{H}_1$, $Q \in \mathcal{H}_2$, and $P \neq Q$, then $\overline{PQ} \cap l \neq \varnothing$.

The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *halfplanes* (or *sides*) of the line l, and l is called an *edge* of each of the halfplanes $\mathcal{H}_1$ and $\mathcal{H}_2$. It is proven in [13] that the halfplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint and nonempty. When quoting the Plane Separation Postulate, we will abbreviate it by PSP.

Given three distinct noncollinear points A, V, and B, then $\angle AVB = \overrightarrow{VA} \cup \overrightarrow{VB}$. We call $\angle AVB$ the *angle* with vertex V and sides $\overrightarrow{VA}$ and $\overrightarrow{VB}$. The *interior* of $\angle AVB$ is the intersection of the halfplane of $\overleftrightarrow{VA}$ that contains B and the halfplane of $\overleftrightarrow{VB}$ that contains A. We denote the interior of $\angle AVB$ by $int(\angle AVB)$.

The following theorem is a consequence of the Plane Separation Postulate, and will be used several times throughout the paper. A proof of this theorem can be found in [8], [13], [14].

Theorem 2.1. (Crossbar) If point P is in the interior of an angle $\angle AVB$, then ray $\overrightarrow{VP}$ intersects segment $\overline{AB}$ at a point T such that $A - T - B$.

The Protractor Postulate:
There exists a function m from the set of all angles to the open interval $(0, \pi)$ such that
(1) For each ray $\overrightarrow{PQ}$ on the edge of halfplane $\mathcal{H}$ (where $\mathcal{H}$ is a halfplane of line $\overleftrightarrow{PQ}$), and for each $r \in (0, \pi)$, there exists a unique ray $\overrightarrow{PR}$, with $R \in \mathcal{H}$, such that $m(\angle QPR) = r$.
(2) If T is a point in the interior of $\angle QPR$, then $m(\angle QPT) + m(\angle TPR) = m(\angle QPR)$.

Given an angle $\angle ABC$, then $m(\angle ABC)$ is called the *measure* of angle $\angle ABC$, and is denote by $m\angle ABC$. Two angles $\angle ABC$ and $\angle DEF$ are said to be *congruent* if $m\angle ABC = m\angle DEF$, in which case we write $\angle ABC \cong \angle DEF$.

3 Side-Angle-Side and Side-Side-Angle

In this section, we give the statements of Side-Angle-Side and Side-Side-Angle.

Given three noncollinear points A, B, and C, then $\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$. We call $\triangle ABC$ a *triangle* with vertices A, B, and C. Given two triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \cong \triangle DEF$ if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that when using the notation $\triangle ABC \cong \triangle DEF$ defined above, the order of the vertices for the triangles $\triangle ABC$ and $\triangle DEF$ is important.

Side-Angle-Side: Given triangles $\triangle ABC$ and $\triangle DEF$, if $AB = DE$, $BC = EF$, and $m\angle ABC = m\angle DEF$, then $\triangle ABC \cong \triangle DEF$. When referring to Side-Angle-Side, we will abbreviate it as *SAS*.

Side-Side-Angle: Let triangles $\triangle ABC$ and $\triangle DEF$ be such that $AB = DE$, $BC = EF$, and $m\angle BCA = m\angle EFD$. Then at least one of the following statements is true:

(i) $\triangle ABC \cong \triangle DEF$

(ii) angles $\angle BAC$ and $\angle EDF$ are supplementary

Furthermore, if $AB \geq BC$, then statement (i) is true. When referring to Side-Side-Angle, we will use the abbreviation *SSA*.

The case where $AB \geq BC$ is often refered to as *Side-side-Angle*, with the middle "side" not capitalized. For the remainder of this paper, we consider both cases (i) and (ii) in Side-Side-Angle, and therefore simply refer to it as Side-Side-Angle. The next theorem examines the case when both statements (i) and (ii) are true simultaneously.

Theorem 3.1. Given triangles $\triangle ABC$ and $\triangle DEF$, assume that $AB = DE$, $BC = EF$, and $m\angle BCA = m\angle EFD$. Then both statements (i) and (ii) are true in SSA if and only if angles $\angle BAC$ and $\angle EDF$ are right angles.

Proof. First assume that both statements (i) and (ii) are true. Since (i) is true, then $\triangle ABC \cong \triangle DEF$, which implies that $m\angle BAC = m\angle EDF$. Since (ii) is true, then $m\angle BAC + m\angle EDF = \pi$. Thus, $m\angle BAC = m\angle EDF = \frac{\pi}{2}$. Hence, $m\angle BAC$ and $m\angle EDF$ are right angles.

Conversely, assume that $m\angle BAC$ and $m\angle EDF$ are right angles. It follows immediately that statement (ii) holds. We need only show that statement (i) holds. If $AB \geq BC$, then we apply SSA to triangles $\triangle ABC$ and $\triangle DEF$ using angles $\angle BCA$ and $\angle EFD$ to get $\triangle ABC \cong \triangle DEF$. If $AB < BC$, then we apply SSA to triangles $\triangle ABC$ and $\triangle DEF$ using angles $\angle BAC$ and $\angle EDF$ to again get $\triangle ABC \cong \triangle DEF$. In either case, we have that $\triangle ABC \cong \triangle DEF$. Hence, both statements (i) and (ii) are true. □

Note that in the case when $AB < BC$, we are essentially using Hypotenuse-Leg to show that $\triangle ABC \cong \triangle DEF$.

We refer the reader to definitions, notation, lemmas, propositions, theorems, and corollaries stated in pages 65 through 194 of [13] which are used throughout this paper. The results proven in pages 65 through 194 of [13] are based only on our initial assumptions and definitions. Also, we refer the reader to definitions and notation introduced in pages 195 through 268 of [13]. There are certain results in pages 195 through 268 of [13] whose proofs are based on our initial assumptions together with Side-Angle-Side. We give alternate proofs of some of these results, with our proofs being based on our initial assumptions together with Side-Side-Angle. In order to state and discuss these results, we need to use definitions and notation stated in pages 195 through 268 of [13]. However, no result from pages 195 through 268 of [13] will be used until it is first proven using Side-Side-Angle.

4 Side-Angle-Side Implies Side-Side-Angle

In this section we show that if we assume Side-Angle-Side, then we can prove Side-Side-Angle. The results in this section are well-known, and are included here for the sake of completeness. In the proof of the following theorem we use several results that depend on SAS, the proofs of which can be found in [13] and [14]. However, in later sections we give proofs of these results that do not depend on SAS, but instead depend on SSA.

Theorem 4.1. Assume that Side-Angle-Side holds. Then Side-Side-Angle holds as well.

Proof. Let triangles $\triangle ABC$ and $\triangle DEF$ be such that $AB = DE$, $BC = EF$, and $m\angle BCA = m\angle EFD$. If $AC = DF$, then it follows by SAS that $\triangle ABC \cong \triangle DEF$, in which case statement (i) of SSA is true.

Assume $AC > DF$. There exists a point T such that $A - T - C$ and $CT = DF$. Applying SAS, we have that $\triangle BCT \cong \triangle EFD$. This implies that $BT = DE = BA$ and $m\angle BTC = m\angle EDF$. By applying the Pons Asinorum to $\triangle ABT$, we have that $m\angle BAT = m\angle BTA$. Since $\angle BTA$ and $\angle BTC$ form a linear pair, then they are supplementary. Thus, $m\angle BAC +$

$m\angle EDF = m\angle BAT + m\angle BTC = m\angle BTA + m\angle BTC = \pi$. Hence, in this case angles $\angle BAC$ and $\angle EDF$ are supplementary, and statement (ii) of SSA is true.

We finally show that if $AB \geq BC$, then statement (i) of SSA is true. Assume $AB \geq BC$, and suppose that $AC > DF$. As above, there exists a point T such that $A - T - C$ and $CT = DF$. Applying SAS, we have that $\triangle BCT \cong \triangle EFD$. This implies that $BT = DE = BA$. However, since $AB \geq BC$, then it must be the case that $BT < AB$. Hence, we have a contradiction and it follows that statement (i) of SSA is true. □

5 Side-Side-Angle Implies Side-Angle-Side

For the remainder of this paper, we show that if we assume Side-Side-Angle, then we can prove Side-Angle-Side. We start by first giving alternate proofs of various results using only SSA. These proofs are independent of SAS.

Theorem 5.1. (Pons Asinorum) Given triangle $\triangle ABC$, if $AB = BC$, then $m\angle BAC = m\angle BCA$.

Proof. By applying SSA to triangles $\triangle ACB$ and $\triangle CAB$, we have that either $\triangle ACB \cong \triangle CAB$ or angles $\angle BAC$ and $\angle BCA$ are supplementary. If $\triangle ACB \cong \triangle CAB$, then it follows that $m\angle BAC = m\angle BCA$.

Suppose that $\angle BAC$ and $\angle BCA$ are supplementary. Let D be a point such that $A - C - D$. Since $\angle BCA$ and $\angle BCD$ form a linear pair of angles, then they are supplementary. Thus, we have that $m\angle BAD = m\angle BAC = m\angle BCD$. If $BD \geq BC$, then using angles $\angle BAD$ and $\angle BCD$, we have by SSA that $\triangle DBA \cong \triangle DBC$. This implies that $DC = DA$, a contradiction since $A - C - D$. If $BD < BC$, then using angle $\angle BDC$, we have by SSA that $\triangle ABD \cong \triangle CBD$, again implying that $DC = DA$, a contradiction. In either case, we get a contradiction, and it follows that $\angle BAC$ and $\angle BCA$ are not supplementary. Hence, $m\angle BAC = m\angle BCA$. □

Let A and B denote distinct points. Let C and D denote points on opposite sides of line $\overleftrightarrow{AB}$. Then we say that angles $\angle ABC$ and $\angle ABD$ form an *adjacent pair* of angles with common side $\overrightarrow{BA}$.

Lemma 5.1. Assume that angles $\angle ABC$ and $\angle ABD$ form an *adjacent pair* of angles. If $\angle ABC$ and $\angle ABD$ are supplementary, then $C - B - D$. In particular, C, B, and D are collinear.

Proof. Let E be a point on the same side of line $\overleftrightarrow{AB}$ as D such that $E - B - C$. Since $\angle ABC$ and $\angle ABE$ form a linear pair of angles, then they are supplementary. Thus, $m\angle ABE = \pi - m\angle ABC$. Since $\angle ABC$ and $\angle ABD$ are supplementary, then $m\angle ABD = \pi - m\angle ABC$. Therefore, we see that $m\angle ABE = \pi - m\angle ABC = m\angle ABD$. It follow by the uniqueness of ray $\overrightarrow{BE}$ in part (1) of the Protractor Postulate that $\overrightarrow{BE} = \overrightarrow{BD}$. Therefore, since $C - B - E$, then $C - B - D$. □

Theorem 5.2. (Converse of the Pons Asinorum) Given triangle $\triangle ABC$, if $m\angle BAC = m\angle BCA$, then $AB = BC$.

Proof. Let M denote the midpoint of segment $\overline{AC}$. If $MB \geq MA$, then it follows by SSA that $\triangle BMA \cong \triangle BMC$. In this case, $AB = BC$. If $MA > MB$, then it follows by SSA that either $\triangle BMA \cong \triangle BMC$ or else that $\angle MBA$ and $\angle MBC$ are supplementary.

Suppose that $\angle MBA$ and $\angle MBC$ are supplementary. This implies that $\angle MBA$ and $\angle MBC$ form an adjacent pair of supplementary angles. Thus, it follows by Lemma 5.1 that $A - B - C$. This contradicts the fact that A, B and C are not collinear. Hence, it follows that $\triangle BMA \cong \triangle BMC$, and consequently that $AB = BC$. □

Theorem 5.3. Given distinct points A and B, then a point P is on the perpendicular bisector of segment $\overline{AB}$ if and only if P is equidistant from A and B.

Proof. First assume that $PA = PB$. Let M denote the midpoint of segment $\overline{AB}$. If $P = M$, then P is on the perpendicular bisector of $\overline{AB}$.

Assume that $P \neq M$, and consequently that A, B, and P are not collinear. By the Pons Asinorum applied to $\triangle PAB$, we have that $m\angle PAB = m\angle PBA$. By SSA, either $\triangle PMA \cong \triangle PMB$ or else $\angle BPM$ and $\angle APM$ are supplementary. Suppose that $\angle BPM$ and $\angle APM$ are supplementary. This implies that $\angle BPM$ and $\angle APM$ form an adjacent pair of supplementary angles. Thus, it follows by Lemma 5.1 that $A - P - B$. This contradicts the fact that A, B and P are not collinear. Thus, $\triangle PMA \cong \triangle PMB$. This implies that $\angle PMA$ and $\angle PMB$ form a linear pair of congruent angles. Thus, $\angle PMA$ and $\angle PMB$ are right angles. Therefore, $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$.

Conversely, assume that P is a point on the perpendicular bisector of segment $\overline{AB}$. Again, let M denote the midpoint of $\overline{AB}$. If $P = M$, then it follows immediately that $PA = PB$. Assume that $P \neq M$, and consequently that the points P, A, and M are not collinear. Suppose that $m\angle PBM > m\angle PAM$. Let D be the unique point on $\overline{PA}$ such that $A - D - P$ and $m\angle DAB = m\angle DBA$. By the converse of the Pons Asinorum applied to triangle $\triangle ABD$, we have that $DA = DB$. Using an argument similar to the one given above, we have that $\overleftrightarrow{DM}$ is the perpendicular bisector of $\overline{AB}$. Since $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$, then it follows that $\overleftrightarrow{DM} = \overleftrightarrow{PM} = \overleftrightarrow{DP}$. Since $A - D - P$, then $\overleftrightarrow{DP} = \overleftrightarrow{DA}$. This implies that the points P, A, and M are on line $\overleftrightarrow{PM}$, a contradiction since they are not collinear. Thus, it cannot be the case that $m\angle PBM > m\angle PAM$. A similar argument shows that we cannot have $m\angle PBM < m\angle PAM$. The only remaining possibility is that $m\angle PBM = m\angle PAM$. Hence, it follows by the converse of the Pons Asinorum applied to triangle $\triangle PAB$, that $PA = PB$. □

Lemma 5.2. Let A, B and P be distinct noncollinear points such that $PA = PB$, and let M be the midpoint of segment $\overline{AB}$. Then ray $\overrightarrow{PM}$ is the angle bisector of $\angle APB$.

Proof. By the Pons Asinorum applied to $\triangle PAB$, we have that $m\angle PAB = m\angle PBA$. By SSA, either $\triangle PMA \cong \triangle PMB$ or else $\angle BPM$ and $\angle APM$ are supplementary. Suppose that $\angle BPM$ and $\angle APM$ are supplementary. This implies that $\angle BPM$ and $\angle APM$ form an adjacent pair of supplementary angles. Thus, it follows by Lemma 5.1 that $A - P - B$. This contradicts the fact that A, B and P are not collinear. Thus, $\triangle PMA \cong \triangle PMB$. This implies that $m\angle BPM = m\angle APM$, which in turn implies that ray $\overrightarrow{PM}$ is the angle bisector of $\angle APB$. □

Theorem 5.4. Given a point P and a line l, then there exists a unique line m passing through P such that m is perpendicular to l.

Proof. If P is on l, then the existence and uniqueness of m follows immediately by the Protractor Postulate. Assume that P is not on l. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the halfplanes of l. We may assume without loss of generality that P is in $\mathcal{H}_1$. Let A and B denote distinct points on l. If $m\angle PAB = \frac{\pi}{2}$, then $\overleftrightarrow{PA}$ is a line through P that is perpendicular to l.

Assume that $m\angle PAB \neq \frac{\pi}{2}$. Let Q be the unique point in $\mathcal{H}_2$ such that $AQ = AP$ and $m\angle BAQ = m\angle BAP$. Since $AQ = AP$, then it follows by Theorem 5.3 that A is on the perpendicular bisector of $\overline{PQ}$. Let M denote the midpoint of $\overline{PQ}$. We assume that B is in the interior of angle $\angle PAQ$. The case where B is not in the interior of angle $\angle PAQ$ is similar, and is left to the reader. By Lemma 5.2, ray $\overrightarrow{AM}$ is the angle bisector of $\angle PAQ$. Since $m\angle PAB = m\angle QAB$, then ray $\overrightarrow{AB}$ is the angle bisector of $\angle PAQ$. Thus, $\overleftrightarrow{AM} = \overleftrightarrow{AB} = l$. Since $\overleftrightarrow{AM}$ is perpendicular to $\overline{PQ}$, then $\overleftrightarrow{PQ}$ is a line through P that is perpendicular to l.

Next we show that the line through P that is perpendicular to l is unique. Suppose that there exist two distinct lines q_1 and q_2 passing through P such that $q_1 \perp l$ and $q_2 \perp l$. Let F_1 and F_2 denote the feet of the perpendiculars q_1 and q_2, resectively. By the converse of the pons asinorum applied to $\triangle PF_1F_2$, we have that $PF_1 = PF_2$. Let D be a fixed point on l such that $F_1 - F_2 - D$. If $PD \geq PF_1$, then using angles $\angle PF_1D$ and $\angle PF_2D$, we see by SSA that $\triangle DPF_1 \cong \triangle DPF_2$. This implies that $DF_1 = DF_2$, a contradiction since $F_1 - F_2 - D$. If $PD < PF_1$, then using angle $\angle PDF_1$, we see by SSA that $\triangle DPF_1 \cong \triangle DPF_2$, again a contradiction. In either case, we get a contradiction and it follows that the line through P that is perpendicular to l is unique. □

Theorem 5.5. Given a point P and a line l not passing through P, then there do not exist distinct points A and B on l such that both angles $\angle PAB$ and $\angle PBA$ are obtuse.

Proof. Let $\mathcal{H}$ denote the halfplane of l containing P. Suppose that there exist distinct points A and B on l such that both angles $\angle PAB$ and $\angle PBA$ are obtuse. By the Protractor Postulate there exists a unique ray $\overrightarrow{BT}$ in $\mathcal{H}$ such that $\angle ABT$ is a right angle. Since $m\angle ABT < m\angle ABP$, then T is in the interior of $\angle ABP$. Thus, by Crossbar $\overrightarrow{BT}$ crosses $\overline{AP}$ at a point K such that $A - K - P$ [8], [13], [14]. Similarly, by the Protractor Postulate there exists a unique ray $\overrightarrow{AG}$ in $\mathcal{H}$ such that $\angle BAG$ is a right angle. Since $m\angle BAG < m\angle BAP = m\angle BAK$, then G is in the interior of $\angle BAK$. Therefore, it follows by Crossbar $\overrightarrow{AG}$ crosses $\overline{BK}$ at a point J such that $B - J - K$. Thus, $\overleftrightarrow{JA}$ and $\overleftrightarrow{JB}$ are two distinct lines through J that are perpendicular to l, a contradiction. Hence, no such points A and B can exist. □

Theorem 5.6. (The Triangle Inequality) Given $\triangle ABC$, then $AB + BC > AC$.

Proof. Suppose that $AC \geq AB + BC$. First assume that $AC > AB + BC$. Let D and E be distinct points on $\overline{AC}$ such that $A - D - E - C$, $AD = AB$, and $CE = CB$. By the Pons Asinorum, $m\angle ABD = m\angle ADB$ and $m\angle CBE = m\angle CEB$. It follows by Theorem 5.4 and Theorem 5.5 that all of the angles $\angle ABD$, $\angle ADB$, $\angle CBE$, and $\angle CEB$ are acute.

Since $\angle ADB$ is acute, then $\angle BDE$ is obtuse. Similarly, since $\angle CEB$ is acute, then $\angle BED$ is obtuse. Thus, applying Theorem 5.5 to $\triangle BDE$ we have a contradiction since both $\angle BDE$ and $\angle BED$ are obtuse. Therefore, it cannot be the case that $AC > AB + BC$.

Now assume that $AC = AB + BC$. Let K be the unique point on segment $\overline{AC}$ such that $A - K - C$, $AK = AB$, and $CK = CB$. By the Pons Asinorum, we have that $m\angle ABK = m\angle AKB$ and $m\angle CBK = m\angle CKB$. Since $\angle AKB$ and $\angle CKB$ form a linear pair of angles, then $\angle AKB$ and $\angle CKB$ are supplementary. This implies that $\angle ABK$ and $\angle CBK$ are supplementary. Thus, $\angle ABK$ and $\angle CBK$ form an adjacent pair of supplementary angles, which implies that $A - B - C$, a contradiction since A, B and C are noncollinear.

In either case, we have a contradiction, and it follows that $AB + BC > AC$. □

In the following two theorems, we let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

Theorem 5.7. Let A, B, and C denote three distinct non-collinear points. For each P on ray $\overrightarrow{AC}$, let $x_P = AP$. Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x_P) = BP$. Then f is continuous.

Proof. Let $\epsilon > 0$. Let P_1 and P_2 be points on ray $\overrightarrow{AC}$ such that $|x_{P_1} - x_{P_2}| < \epsilon$. We may assume without loss of generality that $A - P_2 - P_1$. Thus, $AP_1 = AP_2 + P_2P_1$. This implies that $P_1P_2 = AP_1 - AP_2 \geq 0$. Therefore, $|x_{P_1} - x_{P_2}| = |AP_1 - AP_2| = AP_1 - AP_2 = P_1P_2$, which implies that $P_1P_2 < \epsilon$. By the Triangle Inequality applied to $\triangle BP_1P_2$, we have that $BP_1 + P_1P_2 > BP_2$ and $BP_2 + P_1P_2 > BP_1$. This implies that $BP_2 - P_1P_2 < BP_1 < BP_2 + P_1P_2$, or equivalently that $-P_1P_2 < BP_1 - BP_2 < P_1P_2$. Thus, $|f(x_{P_1}) - f(x_{P_2})| = |BP_1 - BP_2| < P_1P_2 < \epsilon$. Hence, f is continuous. □

Let Q denote a point, and let $r \in \mathbb{R}$ be such that $r > 0$. We define the *circle* $\mathcal{S}$ with center Q and radius r to be the set of all points P such that $QP = r$. We say that a point I is an *interior point* of $\mathcal{S}$ if $QI < r$, and we say that a point E is an *exterior point* of $\mathcal{S}$ if $QE > r$.

The following theorem is referred to as *The Line-Circle Theorem* in [8], [13], [14] and is referred to as *The Secant Theorem* in [12]. A proof of the theorem is given in [12], [13], [14]. We include a proof here to show that the theorem depends only on the Triangle Inequality and the Intermediate Value Theorem, not on Side-Angle-Side. When using the theorem, we will refer to it as The Line-Circle Theorem.

Theorem 5.8. (The Line-Circle Theorem) Let $\mathcal{S}$ denote a circle. Assume that a line l passes through an interior point of P of $\mathcal{S}$. Then l intersects $\mathcal{S}$ at exactly two distinct points D_1 and D_2 such that $D_1 - P - D_2$. More specifically, assume l passes through both an interior point P of $\mathcal{S}$ and an exterior point G of $\mathcal{S}$. Then there exists a unique point R satisfying the following conditions

(i) $P - R - G$

(ii) R is a point of intersection of l and $\mathcal{S}$

Proof. Let Q and r denote the center and radius of $\mathcal{S}$, respectively. Since P is an interior point of $\mathcal{S}$, then $QP < r$.

We first prove the existence of the points D_1 and D_2. Let B be a point on l such that $P \neq B$. For each point Y on ray $\overrightarrow{PB}$, let $x_Y = PY$. Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x_Y) = QY$. It follows immediately from Theorem 5.7 that f is continuous. We also see that P is a point on ray $\overrightarrow{PB}$ such that $f(0) = f(PP) = f(x_P) = QP < r$. By the Ruler Postulate, there exists a point E on ray $\overrightarrow{PB}$ such that $x_E = PE = 2r$. By the Triangle Inequality applied to $\triangle PQE$, we see that $PQ + QE > PE = 2r$. Thus, $f(2r) = f(x_E) = QE > 2r - PQ > 2r - r = r$. Since f is continuous, $f(0) < r$, and $f(2r) > r$, then it follows by the Intermediate Value Theorem that there exists $x_{D_1} \in (0, 2r)$ such that $f(x_{D_1}) = r$. Let D_1 be the point on ray $\overrightarrow{PB}$ such that $x_{D_1} = PD_1$. Thus, $QD_1 = f(x_{D_1}) = r$. This implies that D_1 is a point of intersection of l and $\mathcal{S}$.

Let T be a point such that $T - P - B$. An argument similar to the one just given shows that there exists a point D_2 on $\mathcal{S}$ such that $D_1 - P - D_2$.

We next show that there cannot be distinct points Y_1 and Y_2 on both l and $\mathcal{S}$ such that $P - Y_1 - Y_2$. Suppose that there exist two distinct points Y_1 and Y_2 on both l and $\mathcal{S}$ such that $P - Y_1 - Y_2$. Again, since P is an interior point of $\mathcal{S}$, then $QP < QY_1 = QY_2 = r$. Using angle $\angle QPY_1$, we see by SSA that $\triangle Y_1QP \cong \triangle Y_2QP$. This implies that $PY_1 = PY_2$, a contradiction since $P - Y_1 - Y_2$. Thus, there cannot exist distinct points Y_1 and Y_2 on both l and $\mathcal{S}$ such that $P - Y_1 - Y_2$.

Assume that l passes through both an interior point P of $\mathcal{S}$ and an exterior point G of $\mathcal{S}$. An argument similar to the one just given, with $2r$ replaced by the value x_G, shows the existence of the unique point R satisfying conditions *(i)* and *(ii)*. □

Theorem 5.9. (The Exterior Angle Theorem) Given $\triangle ABC$, let D be a point such that $A - C - D$. Then $m\angle BCD > m\angle BAC$ and $m\angle BCD > m\angle ABC$.

Proof. Suppose that $m\angle BCD \leq m\angle BAC$. If $m\angle BCD < m\angle BAC$, then we can choose a point G such that $B - G - C$ and $m\angle GCD = m\angle GAC$. Thus, we may assume that $m\angle BCD = m\angle BAC$. Let K be the unique point on ray $\overrightarrow{CD}$ such that $CK = AB$. Using angles $\angle BAC$ and $\angle BCK$, we see by SSA applied to triangles $\triangle KBA$ and $\triangle BKC$ that either $\triangle BKC \cong \triangle KBA$ or else $\angle BKC$ and $\angle KBC$ are supplementary angles.

First assume that $\triangle BKC \cong \triangle KBA$. This implies that $m\angle KBC = m\angle BKA = m\angle BKC$ and $m\angle ABK = m\angle BKC$. Putting these equations together, we get that $m\angle ABK = m\angle KBC$. Since $A - C - K$, then C is in the interior of $\angle ABK$, which implies that $m\angle ABK > m\angle KBC$. Thus, we have a contradiction, and it cannot be the case that $\triangle BKC \cong \triangle KBA$.

Next assume that $\angle BKC$ and $\angle KBC$ are supplementary angles. Let T be a point such that $A - C - K - T$. Since $\angle BKC$ and $\angle KBC$ are supplementary angles, and since $\angle BKC$ and $\angle BKT$ are supplementary angles, then $m\angle KBC = m\angle BKT$. From the argument above, we have that $\triangle BKC$ and $\triangle KBA$ are not congruent triangles. If $BK \geq BA$, then it would follow by SSA that $\triangle BKC \cong \triangle KBA$. Therefore, it must be the case that $BK < BA$. Thus, line $\overleftrightarrow{AK}$ passes through the interior point K of the circle $\mathcal{S}$ with center B and radius BA. By the Line-Circle Theorem, there exists a point J such that J is on both $\overleftrightarrow{AK}$ and $\mathcal{S}$, and such that $A - C - K - J$. Since J is on $\mathcal{S}$, then $BJ = AB = CK$.

Using $\angle BKJ$ and $\angle KBC$, we see by SSA applied to triangles $\triangle JBK$ and $\triangle CKB$ that

$\triangle JBK \cong \triangle CKB$. This implies that $m\angle JBK = m\angle CKB$. Since $\angle CKB$ and $\angle BKJ$ form a linear pair of angles, then they are supplementary. However, since $A - C - K - J$, then $\pi > m\angle ABJ = m\angle ABC + m\angle CBK + m\angle KBJ > m\angle CBK + m\angle KBJ = m\angle BKJ + m\angle CKB$, a contradiction. Thus, in either case we have a contradiction and it follows that $m\angle BCD > m\angle BAC$.

If we let Q be a point such that $B - C - Q$, then $\angle BCD$ and $\angle ACQ$ are vertical angles and therefore congruent. Applying the above argument to $\angle ABC$ and $\angle ACQ$, then we get $m\angle ABC < m\angle ACQ = m\angle BCD$. □

The following two theorems are well-known. We include proofs to show that they depend only on the Pons Asinorum and Exterior Angle Theorem, not on Side-Angle-Side.

Theorem 5.10. Given $\triangle ABC$, if $AB < BC$, then $m\angle BCA < m\angle BAC$.

Proof. Let D be the unique point on segment $\overline{BC}$ such that $B - D - C$ and $AB = BD$. By the Pons Asinorum applied to $\triangle ABD$, we see that $m\angle BAD = m\angle BDA$. By the Exterior Angle Theorem applied to $\triangle ADC$, we have that $m\angle BDA > m\angle DCA = m\angle BCA$. Since $B - D - C$, then it follows that D is in the interior of $\angle BAC$. Thus, $m\angle BAD < m\angle BAC$. Combining the above inequalities, we have that $m\angle BAC > m\angle BAD = m\angle BDA > m\angle BCA$. □

Theorem 5.11. Given $\triangle ABC$, if $m\angle BCA < m\angle BAC$, then $AB < BC$.

Proof. If $AB = BC$, then it follows by the Pons Asinorum that $m\angle BCA = m\angle BAC$, a contradiction. If $AB > BC$, then it follows by Theorem 5.10 that $m\angle BCA > m\angle BAC$, again a contradiction. The only remaining possibility is that $AB < BC$. □

Theorem 5.12. Given $\triangle ABC$, assume $BC \geq BA$. Let D be a point such that $A - B - D$. Let K be a point in the interior of $\angle DBC$ such that $BK = BC$. Then segment $\overline{AK}$ crosses segment $\overline{BC}$ at a point T such that $A - T - K$ and $B - T - C$. Consequently, C is a point in the interior of $\angle ABK$.

Proof. Since K is in the interior of $\angle DBC$, then K and D are on the same side of $\overleftrightarrow{BC}$. Since $A - B - D$, then A and D are on opposite sides of $\overleftrightarrow{BC}$. Thus, A and K are on opposite sides of $\overleftrightarrow{BC}$. By PSP, $\overline{AK}$ crosses $\overleftrightarrow{BC}$ at a point T such that $A - T - K$.

First suppose that $T = C$. By the Pons Asinorum applied to $\triangle CBK$, we have that $m\angle BCK = m\angle BKC$. By the Exterior Angle Theorem applied to $\triangle ABC$, we have that $m\angle BAC < m\angle BCK = m\angle BKC$. By applying Theorem 5.10 to $\triangle ABK$, we have that since $BK = BC \geq BA$, then $m\angle BAC \geq m\angle BKC$. Thus, we have a contradiction and it cannot be the case that $T = C$.

Next suppose that $T = B$. This implies that $A - B - K$. However, since K is in the interior of $\angle DBC$, then A, B, and K are not collinear, a contradiction. Thus, $T \neq B$.

Suppose that $T - B - C$. In this case, T and K are on opposite sides of $\overleftrightarrow{AD}$. By PSP, there exists a point W such that $T - W - K$. If $W = A$, then we have both $T - A - K$ and $A - T - K$, which cannot happen simultaneously. Thus, $W \neq A$. However, this implies that distinct lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{AK}$ have the two distinct points A and W in common, a contradiction.

Suppose that $B-C-T$. In this case, we have that $BT > BC = BK$. Since K is in the interior of $\angle DBC$, then B, K, and T are not collinear. By applying Theorem 5.10 to $\triangle BKT$, we have that $m\angle BKT > m\angle BTK$. By the Exterior Angle Theorem applied to $\triangle ABT$, we have that $m\angle BTK > m\angle BAT$. Thus, $m\angle BKT > m\angle BTK > m\angle BAT$. Since $BK = BC \geq BA$, then it follows by Theorem 5.10 applied to $\triangle ABK$ that $m\angle BAT \geq m\angle BKA = m\angle BKT$. Thus, we have a contradiction, and it cannot be the case that $B-C-T$.

The only remaining possibility is that $B-T-C$. It follows immediately that both points T and C are in the interior of $\angle ABK$. □

Theorem 5.13. Given isosceles triangle $\triangle ABC$ such that $AB = BC$, let D be a point such that $A-D-C$. Then $BD < AB$.

Proof. Suppose that $BD \geq AB$. First assume that $BD = AB$. In this case, we have the three isosceles triangles $\triangle ABC$, $\triangle ABD$, and $\triangle DBC$. By the Pons Asinorum applied to $\triangle ABC$ and $\triangle DBC$, we have that $m\angle BAC = m\angle BCA = m\angle BDC$. However, this violates the Exterior Angle Theorem applied to $\triangle ABD$. Thus, we cannot have $BD = AB$.

Next assume that $BD > AB$. By the Pons Asinorum applied to $\triangle ABC$, we have that $m\angle BAC = m\angle BCA$. Since $BD > AB = BC$, then it follows by Lemma 5.10 that $m\angle BCD > m\angle BDC$. Thus, $m\angle BAD = m\angle BCD > m\angle BDC$. This violates the Exterior Angle Theorem applied to $\triangle ABD$. Thus, we cannot have $BD > AB$. In either case, we get a contradiction, and the only remaining possibilty is that $BD < AB$. □

Assume that $\triangle ABC$ is such that $BC \geq AB$. Let D be a point such that $A-B-D$. Let $\mathcal{S}$ denote the circle with center B and radius BC. Let K be a point such that $K \in int(\angle DBC)$, and such that $BK = BC$. By Theorem 5.12, there exists a point T such that $A-T-K$ and $B-T-C$. Let Q be a point such that $B-K-Q$, and such that KQ is sufficiently small so that $m\angle KAQ < m\angle KAC$. Since $B-T-C$ and $A-T-K$, then B and C are on opposite side of line $\overleftrightarrow{AK}$. Since $B-K-Q$, then B and Q are on opposite sides of line $\overleftrightarrow{AK}$. Thus, C and Q are on the same side of line $\overleftrightarrow{AK}$. Since C and Q are on the same side of line $\overleftrightarrow{AK}$, and since $m\angle KAQ < m\angle KAC$, then it follows by part (ii) of the Protractor Postulate that $Q \in int(\angle CAK)$. Thus, it follows by Crossbar that ray $\overrightarrow{AQ}$ crosses segment $\overline{CK}$ at point J such that $C-J-K$. Since $B-K-Q$, then $BK < BQ$. Since $\triangle CBK$ is an isosceles triangle and since $C-J-K$, then it follows by Theorem 5.13 that $BJ < BC = BK < BQ$. Thus, it follows by the Line-Circle Theorem applied to $\mathcal{S}$ that there exists a point P such that $J-P-Q$ and $BP = BC$. In the next theorem, we prove that any point R such that $P-R-Q$ is an exterior point of the circle $\mathcal{S}$.

Theorem 5.14. Given triangle $\triangle ABC$, assume that $BC \geq AB$. Let D be a point such that $A-B-D$. Let $\mathcal{S}$ denote the circle with center B and radius BC. Let K be a point such that $K \in int(\angle DBC)$, and such that $BK = BC$. Let Q be a point such that $B-K-Q$, and such that KQ is sufficiently small so that $m\angle KAQ < m\angle KAC$. Let J be the point of intersection of ray $\overrightarrow{AQ}$ with segment $\overline{CK}$, and let P be the point given by the Line-Circle Theorem on $\mathcal{S}$ such that $J-P-Q$. Then for each point R such that $P-R-Q$, we have that $BR > BC$. That is, R is an exterior point of $\mathcal{S}$.

Proof. Suppose that $BR \leq BC$. First assume that $BR = BC$. In this case, R is a point on line $\overleftrightarrow{JQ}$ distinct from P such that $BR = BC = BP$. Since $J - P - Q$ and $P - R - Q$, then $J - P - R - Q$. Since $BJ < BC$ and $BQ > BC$, then this contradicts the uniqueness of the point P guaranteed by the Line-Circle Theorem. Thus, it cannot be the case that $BR = BC$.

Next assume that $BR < BC$. Since $BR < BC$ and $BQ > BC$, then it follows by the Line-Circle Theorem applied to $\mathcal{S}$ that there exists a point H on line $\overleftrightarrow{JQ}$ such that $R - H - Q$ and $BH = BC = BP$. Since $J - P - Q$, $P - R - Q$, and $R - H - Q$, then $J - P - R - H - Q$. This again contradicts the uniqueness of the point P guaranteed by the Line-Circle Theorem.

In either case, we have a contradiction and it follows that $BR > BC$.

□

Theorem 5.15. Given triangle $\triangle ABC$, assume that $BC \geq AB$. Let D be a point such that $A - B - D$. Let $m\angle BAC = \theta \in (0, \pi)$, and let $m\angle DBC = \phi \in (0, \pi)$. For each $\sigma \in (0, \phi]$, let K_σ be the unique point on the same side of line $\overleftrightarrow{AB}$ as C such that $m\angle DBK_\sigma = \sigma$ and $BK_\sigma = BC$. Define $f : (0, \phi] \to (0, \theta]$ by $f(\sigma) = m\angle BAK_\sigma$. Then

(i) The function f is continuous.

(ii) For each $\alpha \in (0, \theta)$, there exists $\sigma \in (0, \phi)$ such that $m\angle BAK_\sigma = \alpha$. That is, given $\alpha \in (0, \theta)$, there exists a point $K \in int(\angle DBC)$ such that $m\angle BAK = \alpha$ and $BK = BC$.

(iii) For each $\sigma \in (0, \phi)$, $C \in int(\angle ABK_\sigma)$.

Note that in Theorem 5.15, $K_\sigma = C$ if $\sigma = \phi$, and $K_\sigma \in int(\angle DBC)$ if $\sigma < \phi$.

Proof. We first prove *(i)*. Let $\epsilon > 0$. Let $\lambda \in (0, \phi]$. We may assume that $0 < \epsilon < m\angle CAK_\lambda$. Let F be a point on $\overleftrightarrow{BK_\lambda}$ such that $B - K_\lambda - F$ and $m\angle FAK_\lambda < \epsilon < m\angle CAK_\lambda$. By Theorem 5.12, there exists a point T such that $B - T - C$ and $A - T - K_\lambda$. Since $B - K_\lambda - F$, then B and F are on opposite sides of line $\overleftrightarrow{AK_\lambda}$. Since $A - T - K_\lambda$ and $B - T - C$, then B and C are on opposite sides of line $\overleftrightarrow{AK_\lambda}$. Thus, F and C are on the same side of line $\overleftrightarrow{AK_\lambda}$. Since F and C are on the same side of line $\overleftrightarrow{AK_\lambda}$ and since $m\angle FAK_\lambda < m\angle CAK_\lambda$, then it follows by part (ii) of the Protractor Postulate that $F \in int(\angle CAK_\lambda)$.

Let Q be a point such that $K_\lambda - Q - F$. This implies that Q is a point on line $\overleftrightarrow{BK_\lambda}$ such that $Q \in int(\angle CAK_\lambda)$. We also see that $m\angle QAK_\lambda < m\angle FAK_\lambda < \epsilon$, and that $BQ > BK_\lambda = BC$. Since $Q \in int(\angle CAK_\lambda)$, then it follows by Crossbar that ray $\overrightarrow{AQ}$ crosses segment $\overline{CK_\lambda}$ at a point J such that $C - J - K_\lambda$.

Since $\triangle BCK_\lambda$ is isosceles, then it follows by Theorem 5.13 that $BJ < BC$. Let $\mathcal{S}$ denote the circle with center B and radius BC. Applying the Line-Circle Theorem to $\mathcal{S}$, we see that there exists a point P such that $J - P - Q$ and $BP = BC$. Since J is on ray $\overrightarrow{AQ}$, and since $J - P - Q$, then all three points J, P, and Q are on ray $\overrightarrow{AQ}$. This implies that $m\angle PAK_\lambda = m\angle QAK_\lambda < \epsilon$. Let $\delta = m\angle PBK_\lambda$.

Let W be a point in the interior of $\angle DBC$ which is on the same side of line $\overleftrightarrow{BK_\lambda}$ as P such that $BW = BC$ and such that $m\angle WBK_\lambda < \delta = m\angle PBK_\lambda$. It follows by part (ii) of the Protractor Postulate that $W \in int(\angle PBK_\lambda) = int(\angle PBQ)$. By Crossbar, there exists a point R on ray

$\overrightarrow{BW}$ such that $P - R - Q$. By Theorem 5.14, we have that $BR > BC = BW$, which implies that $B - W - R$. Since $W \in int(\angle PBK_\lambda)$, then there exists a point Z such that $P - Z - K_\lambda$ and $B - Z - W$. In particular, $Z \in int(\angle PAK_\lambda)$. Since $B - Z - W$ and $B - W - R$, then $B - Z - W - R$. Since all three points P, Q, and R are on ray $\overrightarrow{AQ}$, since $Z - W - R$, and since $Z \in int(\angle PAK_\lambda)$, then $W \in int(\angle PAK_\lambda)$. This implies that $m\angle WAK_\lambda < m\angle PAK_\lambda < \epsilon$. Thus, if W is a point in the interior of $\angle DBC$ which is on the same side of line $\overleftrightarrow{BK_\lambda}$ as P such that $BW = BC$ and such that $m\angle WBK_\lambda < \delta$, then $m\angle WAK_\lambda < \epsilon$.

By the argument just given, we have that for any $\epsilon > 0$, there exists a $\delta > 0$ with the property that for each $\sigma_1, \sigma_2 \in (0, \phi]$ such that $\sigma_1 > \sigma_2$, if $|\sigma_1 - \sigma_2| = |m\angle DBK_{\sigma_1} - m\angle DBK_{\sigma_2}| = |K_{\sigma_1}BK_{\sigma_2}| < \delta$, then $|f(\sigma_1) - f(\sigma_2)| = |m\angle BAK_{\sigma_1} - m\angle BAK_{\sigma_2}| = |K_{\sigma_1}AK_{\sigma_2}| < \epsilon$. Hence, f is continuous.

We next prove *(ii)*. Let $\alpha \in (0, \theta)$, and let $\hat{\sigma} = \frac{1}{2}\alpha$. By the Exterior Angle Theorem applied to $\triangle ABK_{\hat{\sigma}}$, we have that $f(\hat{\sigma}) = m\angle BAK_{\hat{\sigma}} < m\angle DBK_{\hat{\sigma}} = \hat{\sigma} = \frac{1}{2}\alpha < \alpha$. By definition of f, we have that $f(\phi) = \theta > \alpha$. Since f is continuous, then it follows by the Intermediate Value Theorem that there exists $\sigma \in (\hat{\sigma}, \phi) \subseteq (0, \phi)$ such that $m\angle BAK_\sigma = f(\sigma) = \alpha$.

Finally, we prove *(iii)*. Let $\sigma \in (0, \phi)$. Then C and K_σ are on the same side of line $\overleftrightarrow{AB}$. Since $A - B - D$, then angles $\angle ABC$ and $\angle DBC$ are supplementary. Similarly, angles $\angle ABK_\sigma$ and $\angle DBK_\sigma$ are supplementary. Since $\sigma \in (0, \phi)$, then $\pi - m\angle ABK_\sigma = m\angle DBK_\sigma = \sigma < \phi = m\angle DBC = \pi - m\angle ABC$. Thus, $m\angle ABK_\sigma > m\angle ABC$. Since C and K_σ are on the same side of line $\overleftrightarrow{AB}$, then it follows by part (ii) of the Protractor Postulate that $C \in int(\angle ABK_\sigma)$. □

We are now ready to prove Side-Angle-Side.

Theorem 5.16. (Side-Angle-Side) Given triangles $\triangle ABC$ and $\triangle DEF$, if $AB = DE$, $BC = EF$, and $m\angle ABC = m\angle DEF$, then $\triangle ABC \cong \triangle DEF$.

Proof. We may assume without loss of generality that $BC \geq AB$. If $m\angle BAC = m\angle EDF$, then it follows by SSA that $\triangle ABC \cong \triangle DEF$. Suppose that $m\angle BAC < m\angle EDF$. By Theorem 5.15 applied to triangle $\triangle DEF$, there exists a point K such that $EK = EF = BC$, $m\angle BAC = m\angle EDK$, and $F \in int(\angle DEK)$. Thus, it follows by SSA that $\triangle ABC \cong \triangle DEK$. This implies that $m\angle ABC = m\angle DEK$. Since $F \in int(\angle DEK)$, then it follows by the Protractor Postulate that $m\angle ABC = m\angle DEF < m\angle DEK$, a contradiction. A similar argument shows that it cannot be the case that $m\angle BAC > m\angle EDF$. Therefore, it must be that case that $m\angle BAC = m\angle EDF$. Hence, $\triangle ABC \cong \triangle DEF$. □

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A Mathematical Taorist in Göttingen

Johan Gielis

Abstrakt
Matematické centrá,
kvitnú a blednú,
matematika samotná
nikdy nebude slúžiť twitteru.
Algebra, analýza a geometria,
Svätá Trojica,
božská podstata matematiky.
Vrátiť sa v čase ako matematický taorista.
Ustanoviť, čo je zakorenené v jazyku,
v aoristovi.

Kľúčové slová: dejiny matematiky, geometria, zovšeobecnené Möbiove-Listingove telesá, aorista

Abstract
Mathematical centers,
they bloom and they whither,
Mathematics itself
will never be servant to twitter.
Algebra, calculus & geometry,
A Holy Trinity
Mathematics' own Divinity
Going back in time as a Mathematical Taorist
To establish it is rooted in language,
in the Aorist

Keywords: history of mathematics, geometry, generalized Möbius-Listing bodies, aorist

1 Walking on sacred soil

In November 2018 I drove to the University of Göttingen in Lower Saxony, Germany for a visit I had long looked forward to. In the 19th century up to 1933, Göttingen was one of the foremost mathematical powerhouses in the world. Illustrious mathematicians like Carl-Friedrich Gauss, a.k.a. the Prince of Mathematics, Bernhard Riemann, Gustave Dirichlet, Felix Klein, Emmy Noether, David Hilbert, Hermann Minkowski, and many others who walked the streets of Göttingen, were members of the Göttingen Academy or spent their lives in the Göttingen observatory.

The purpose of my visit was twofold; first, to attend a workshop on Sparse Approximation with Exponential Sums and Applications, organized by Gerlind Plonka-Hoch of the Institute for Numerical and Applied Mathematics, and secondly, to visit a colleague in the Department of Ecology and Evolution, both at the University of Göttingen (in German the Georg-August-Universität Göttingen).

During the workshop I discussed my generalization of Lamé curves, which in this sense allows for extremely sparse shape descriptors, in comparison with finite or infinite series expansions [1]. I took this opportunity to highlight the great mathematical tradition of Göttingen; my work is in many ways both a continuation and a tribute to this legacy. Mathematics in Göttingen was, while pure in the purest sense, often motivated by natural sciences and natural philosophy, mainly physics and astronomy (Fig. 1). My background in botany and my research in plants allowed me to generalize Pythagoras' Theorem to what are now called Gielis curves, surfaces and transformations, providing a unifying description of natural shapes, at all levels [2]. This generalization might have been known already in the early 1800's, in Göttingen or Paris, but it was only published in the last years of the 20th century. This unifying description for a wide variety of natural shapes is based on observations in plants, and for this I was called "A botanical Kepler, awaiting his Newton" [3].

Fig. 1. Möbius band in front of the Göttingen Observatory

My journey in mathematics took off after my initial publications, and led to a variety of collaborations. Coming from the world of botany, I soon learned about the extended networks, the historical lineages and the strong oral tradition in geometry. What one finds only sporadically in writing are the big open challenges, and these are not necessarily the Millenium Prize ones or Hilbert's 23 problems. What ends up in journals, books and textbooks are "finished products", the theorems and proofs, the exercises with answers. Many of the open questions however, are transmitted in oral tradition mainly, from master to student, through generations.

Another important lesson was that progress is only slow in mathematics and science. Between important steps, decades, centuries or eons may pass. What is considered progress for some has by no means the same meaning for others. As Sir Michael Atiyah recounts [4], Arnold's "*view of mechanics, in fact of physics, is fundamentally geometrical, going back to Newton. Everything in between, with the exception of a few people like Riemann, who was a bit of a digression, was a mistake*". What Gauss and Riemann, among all the fantastic work they published, were really looking for was a unifying description of nature, in the sense of Kepler and Newton. This was also the dream of many of their predecessors and their descendants. Despite the many applications of mathematics in biology, no mathematical model or system can describe the evolution of individual lives and feelings. We are only at the beginning of understanding nature.

One example of the oral tradition is non-Euclidean geometry, famously discovered independently by Nikolaj Lobachevsky, Farkas Bolyai and Carl Friedrich Gauss. The motivations trace back to Göttingen, with leading roles for Martin Bartels, Johann Pfaff and perhaps Abraham Kästner. Lobachevsky graduated at the University of Kazan under supervision of Bartels, but when he was young, Bartels was also the tutor of young Gauss. Pfaff was Gauss' mentor, and Gauss and Farkas Bolyai, father of Janos Bolyai, were close friends in Göttingen. The idea to dismiss the parallel postulate in Euclidean geometry was out in the open in Göttingen.

Many of the developments in contemporary mathematics can be traced back to the 19th century, to Göttingen. To name only one: twentieth century geometry was dominated by Riemannian geometry, while twenty first century geometry will be dominated by Riemann-Finsler geometry [5]. This originates in a side remark of Riemann on using other than quadratic forms in his famous Habilitationsvortrag. Only half a century later Paul Finsler, in his thesis with Constantin Carathéodory, started the study of the more general ways of measuring. One of the simplest

Riemann-Finsler geometries is based on the unit circle of Lamé type, i.e. $x^n + y^n = R^n$, geometrical equivalent of the Last Theorem of Fermat, and the starting point of my own work [2].

Numerous genealogies show the connection of Göttingen to many of the most important mathematical centres of the world today. Saint-Petersburg is one example, and a famous mathematician in Saint-Petersburg was Pafnuty Chebyshev, who graduated with Lobachevsky. This led to lineages with the Markov brothers, Aleksandr Lyapunov, Alexander Friedmann, A. N. Alexandrov and G. Perelman. Another lineage is the Moscow school, founded by Egorov and Bugaev, with mathematical offsprings like Kolmogorov, Gelfand, Arnold, Petrovski and Oleinik. The mathematical great-grandfather of John Nash is Felix Klein. The role of Göttingen as mathematical centre abruptly stopped in 1933 when the Nazi's came to power, but its legacy was secured. Göttingen's best mathematicians were welcomed in the USA.

2 Torsion of prisms

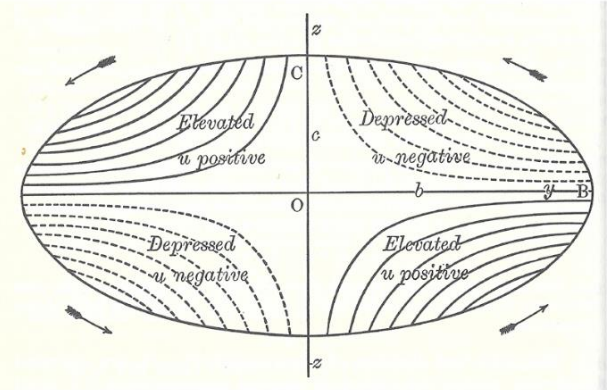
One of the highlights of my visit was the mathematical models collection at the Georg-August-Universität. It consists of a vast collection of cardboard, plaster, metal and wooden models, to aid students and mathematicians in understanding concepts and functions. The first models in the collection were cardboard models developed under the supervision of Abraham Kästner from 1780 onwards. Various models were featured in Hilbert and Cohn-Vossen's *Anschauliche Geometrie* [6], translated in English as *Geometry and the Imagination*.

A virtual tour of the collection is possible via
https://www.math.uni-goettingen.de/historisches/modelcollection.html

During the guided tour, I encountered some 'unknown' gypsum models (Fig. 2). These were unlabeled and their origin and meaning were unknown, according to our guide professor Samuel Patterson, who has been curator of the collection from 1982 to 2011. I have recognized these models immediately: they were models of Saint-Venant's research on torsion of prisms. Adhémar Jean Claude Barré de Saint-Venant (1797-1886) was a contemporary of Gabriel Lamé (1795-1860).

Fig. 2. Unlabelled models in Göttingen collection

If a cylinder is subjected to torsion, the circular cross sections will remain flat. The research of Saint-Venant on the torsion of prisms showed how the internal body is subjected to stress resulting in distorted cross sections, when the cross sections are not circular, as in prisms or non-circular rods. In Isaac Todhunter's *The history of elasticity* [7] Saint-Venant's derivations are described in detail with pictures that are 2D versions of the gypsum models in the Göttingen collection. In Fig. 3 the results are shown for prisms with square and triangular cross sections. The plaster models show the real three-dimensional deformations. I could not observe all the numbers, but in the collection the square one is N°595, the ellipse is N°596, the triangle is N°597 and the star with four rounded points is N°598.



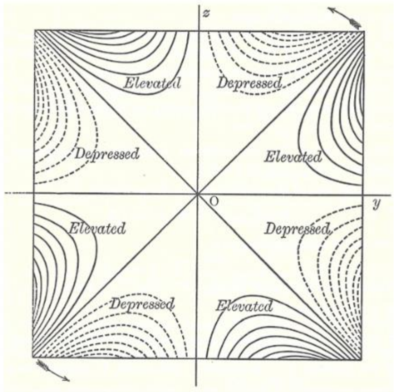


Fig. 3. Torsion of prisms with square and triangular cross-sections.
Dotted lines are depressed regions and full lines are elevated parts of the cross sections [7]

In the exposition there were a few special shapes, which I had not seen before, such as the central one N°598 in Fig. 2. However, these can be found in Isaac Todhunter's A History of the Theory of Elasticity, and they are solutions for prisms with special cross sections studied by Saint-Venant. In Fig. 4 the star with four rounded points is model N°598 in the collection. The square with acute angles in Fig. 4 may also be present, but through the glass windows of the cabinets I could not distinguish whether the model was a square or a square with acute angles.

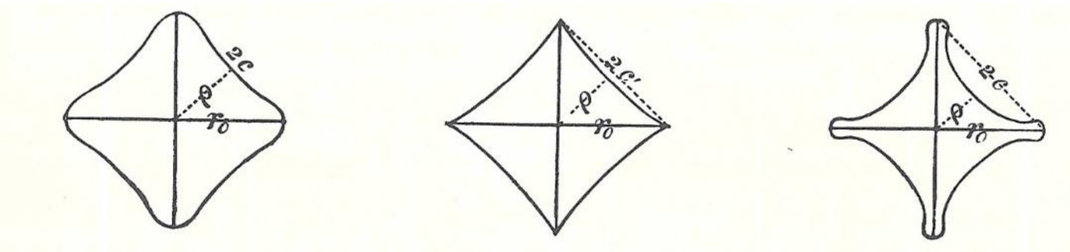


Fig. 4. From left to right: square with rounded angle, square with acute angle (both fourth order), and star with four rounded points (eighth order) [7]

The next question is when and where the models were made and by whom. Interestingly, models N°846 and N°847 in the collection, displaying 3D effects of vibrations, were actually made by Saint-Venant. The notes with N°846 and N°847 state that Saint-Venant provided both models in 1852 to the Société Philomathique de Paris, established in 1788. Given the fact that Saint-Venant was a model maker for mathematical models with gypsum plaster or plaster of Paris, a reasonable hypothesis is that models in Fig. 2 could also be attributed directly to him.

It would be interesting to investigate how they ended up in the Göttingen. They may share a common history with N°846 and N°847: manufactured by Saint-Venant, donated to the Société, then to Göttingen. An inquiry with the Société might clarify this, but a more direct and conclusive evidence of a common origin could be obtained by a simple comparative chemical analysis of one of the models N°s 595-598 and of the models N°846 and N°847.

3 Saint-Venant's Legacy

The importance of the research of Saint-Venant cannot be overestimated, and his results inspired many in the past 150 years. In D'Arcy Thompson's 1917 *On Growth and Form* [8], a wonderful book applying many fundamental 19th century results in geometry and mathematical physics to living organisms, one can find a discussion of Saint-Venant's results in relation to the formation of horns. After remarking that natural shapes are not accurate triangles, in the section The shape of horns of sheep and goats he writes:

"*The better to illustrate this phenomenon, the nature of which is indeed obvious enough from a superficial examination of the horn, I made a plaster cast of one of the horny rings in a horn of Ovis Ammon, so as to get an accurate pattern of its sinuous edges and then, filling the mould up with wet clay, I modelled an anticlastic surface, such as to correspond as nearly as possible with the sinuous outline. Finally, after making a plaster cast of this sectional surface, I drew its contour-lines (as shown in Fig. 322), with the help of a simple form of spherometer. It will be seen that in great part this diagram is precisely similar to Saint-Venant's diagram of the cross section of a twisted triangular prism*".

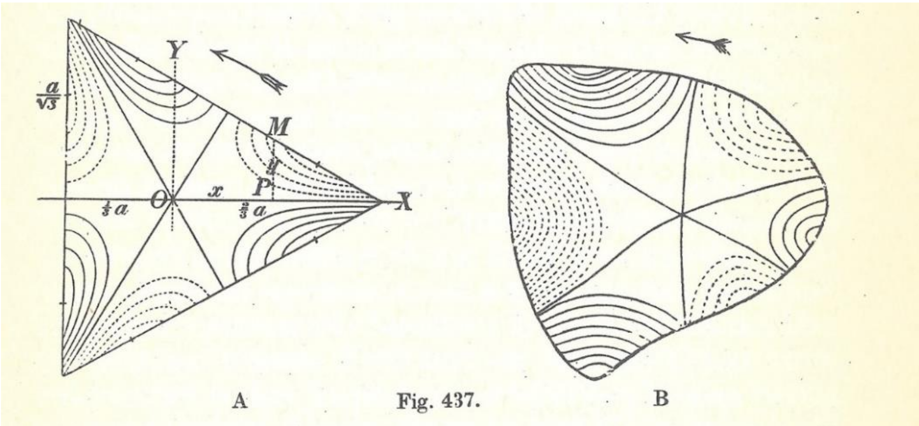


Fig. 5. Figs 321 and 322 from [8]. Fig. 321 is Saint-Venant's triangular cross section

Our own research is related to Saint-Venant and to mathematical research in Göttingen, in particular to Möbius and Listing's non-orientable surfaces [9]. In the 1950 and 1960's, Olga Oleinik in Moscow observed that the solution of boundary value problems would greatly benefit from the knowledge of the domain. My colleague Ilia Tavkhelidze from Tbilisi, then a doctoral student of Oleinik, started to generalize Möbius-Listing bodies and surfaces, and studied the result of their cutting [10]. Hitherto, mainly the classical Möbius ribbons with cross section a line were studied, but in our joint work the generalization allows for cross sections of different planar curves or disks, leading to surfaces or bodies, respectively. Their cutting leads to a direct link with the theory of knots and links [10].

Fig. 6 displays a Generalized Möbius-Listing body with basic line, a circle, and a regular pentagon as cross section; the toroidal structure is twisted a certain number of times. If a cut is made from side 1 to side 3 (counting clockwise) and the knife follows the path until it returns to its original position, the structure will result in four different complex, but yet coherent structures. The different bodies will have cross sectional shapes according to the coloured zones (indicated in yellow, brown, grey or blue), whereby the number of twists of each of the resulting bodies is determined by the original number of twists. The resulting bodies are intricately intertwined (Fig. 6 shows only one of the bodies on the right).

When the knife cuts the center of the polygon, only one body may result, displaying the Möbius phenomenon, depending on the type of cutting [11], [12]. Obviously, in Generalized Möbius-Listing bodies with regular-polygons as cross section, the whole structure is subject to torsion and bending. Combining and generalizing the work of Möbius, Listing and of Saint-Venant will contribute to understand the heart and its action, twisted fiber bundles, DNA and RNA and their operations, among others [13].

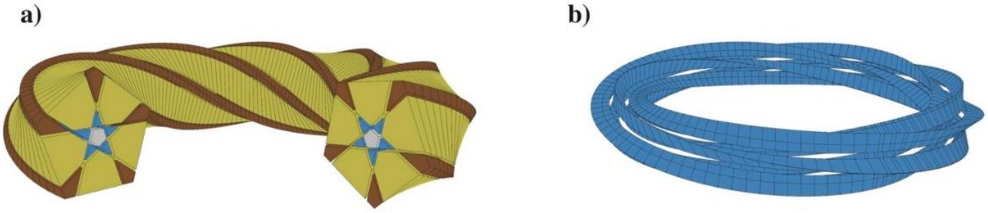


Fig. 6. Generalized Möbius-Listing body and one result after cutting from side to side.

4 An Indo-European origin for mathematics

Another highlight of the collection are the metal wireframes used by Richard Courant. These frames are designed for dipping into soap solutions. Soap films are physical examples of solutions to Plateau’s problem. This is a fantastic way to understand minimal surfaces, and crucial mathematical concepts [14]. How subjects are taught is part of mathematics’ great oral tradition, based on language, both visual and verbal. In *On Proof and Progress* [15], William Thurston describes how people develop “understanding of mathematics”: “*Human language is a pivotal part to understanding mathematics. Intuition, associations, logic, deduction, spatial awareness and communication play important roles. The mathematical language of symbols is closely tied to our human language facility.*”.

Then my mind drifts off to the origin of mathematics. I am of the opinion that language is of primordial importance to understand anything in human culture, including mathematics. Since I started with mathematics (and that was only in my thirties) I have always wondered why language and mathematics are completely separated in thought and teaching, even in our high school system with Latin and Greek. The fundamental contributions of the Greeks to mathematics and natural philosophy on the one hand, and to literature, philosophy, logic, drama and poetry on the other, are clear and unchallenged. In ancient Greece these subjects were strongly intertwined, with Pythagoras’ teaching that *All is number* and Plato’s *Timaeus* and *Theaetetus*. Building a convincing argument in mathematics or before a court relies on the same structure of reasoning and in both cases language is crucial. It is often said that Greek mathematics lacked symbols, which is not true. On the contrary, an additional advantage of Greek mathematicians was that they mastered language. This leads us to the older origin of mathematics.

Besides advancing mathematics and the natural sciences, in Göttingen there was a great interest both in the foundations and the history of mathematics. One of the most important historians of mathematics and astronomy in the first half of the 20th century was Otto Neugebauer, a student of Courant and Hilbert, who focused mainly on Babylonian and Egyptian astronomy. Another student of Courant was Bartelt Leendert Van der Waerden, with great interest in the history and origin of mathematics. After his doctorate in 1926 at the University of Amsterdam, he passed his Habilitation at Göttingen University under the wings of Richard Courant in 1927.

He became Courant's assistant and Privatdozent at Göttingen and later made fundamental contributions to the understanding of Greek mathematics.

In *Geometry and Algebra in Ancient Civilizations* [16], the main thesis is that the origin of mathematics is found in Indo-European history. Van der Waerden noted many similarities between the mathematical and religious ideas present in the Neolithic age, in Greece, India, China, England,…., including the knowledge of Pythagorean triples. Moreover, "*The Indo-European languages are connected with a perfect decimal counting system, including a method for fractions. The religions of ancient Indo-European populations have so much in common that the existence of an Indo-European religion can hardly be doubted. Hence, if we find quite similar ideas about the ritual importance of geometrical constructions in Greece and India, and the same set of Pythagorean triangles with ritual applications in England and India, and the same geometrical constructions in Greece and India, the conclusion that these religious and mathematical ideas have a common Indo-European origin is highly probable*" [16].

As Van der Waerden argues, the mathematical sciences were already well developed by 500 BC, but the Pythagoreans transformed $\mu\alpha\vartheta\eta\mu\alpha\tau\alpha$ (mathemata) into an ordered system with statements and proofs, making it a purely deductive science. He showed that Greek mathematics was indeed advanced in symbols.

5 Beyond space and time: the aorist

This intersects with my own investigations. Whereas language is considered the opposite of mathematics with respect to clarity (think poetry, theatre and storytelling versus axioms, theorems and proofs), mathematics definitely originated in language. The origin of mathematics is tied to the transition from oral traditions to writing and to the simplification of language. This simplification probably happened because of the migrations of Indo-European people to West and East, as far as India and China (extinct Tocharian is of Indo-European origin [16]). One of these simplifications was the transition of different notions of time, to linear time. In my opinion it is precisely this loss in language that became crucial to the development of mathematics as a deductive science.

Time and how it is perceived is a key concept to understanding ancient writings, including Homer or Genesis. In our current science and linear time we distinguish between past, present and future, and in language we add perfect and imperfect forms. While we are convinced of the correctness of our views, Ancient Greek tourists visiting our era might consider us as prisoners of time, since the notion of (linear) time was of only secondary importance to the Greeks. For them it was not important when an action took place, but how. In ancient Indo-European languages and in ancient Greek this is expressed with the aorist.

The aorist form indicates the actions, simple and pure, as if it was a singular point, with beginning and end into one. Neither the space nor time where and when the action took place is important, but the action or process has a definite endpoint. If the imperfect is an indefinite line along which the action unfolds, the aorist is a point or a definite line. For modern scientific human beings, fully accustomed to clocks and the linear time scale, it is almost impossible to understand this. It is however impossible to understand ancient texts such as Genesis, Homer or even Euclid, without understanding the aorist aspect. There are nuances: the complexive aorist concentrates on the whole action and how the action develops into one singular point, with beginning and end. The ingressive aorist focuses more on the starting point, whereas the resultative aorist gives more weight to the end point [17]).

In the oral tradition, the narrator tells a story, placing what occurred before the actual story (and is necessary to understand the story) in aorist aspect. For example in *Genesis* the creation of the world (the action) is in six days (the aorist is used with definite numbers [17]), with basically the correct sequence of how it unfolds. The narrator places the genesis of the universe outside space and time. Likewise, what was before the Big Bang is not relevant, since everything (past-present-future) was already there in Lemaître's *Atome primitif* or cosmic egg [2]. It is all there, at once, as a given, providing for a strong foundation of what is to come. In mathematics: Think axioms and definitions. The aorist states that something happened, and leaves it to the reader to draw consequences of this, in the understanding that what happens once is typical of what happens often (Is mathematics not the science of patterns?) The gnomic aorist speaks of general truths, about main facts that happened. In mathematics: Think theorems and constructing proofs.

Moreover, *αοριστος* (aoristos) means undefined. What is true for one rectangular triangle is true for all rectangular triangles, anywhere, anytime, not for one particular rectangular triangle only. The root of mathematics is the Proto-Indo-European *μανθανω* (manthano), which means learning, knowledge. In Ancient Greek manthano related to searching, seeking, investigating (respectively, 'I learn', 'I seek' I inquire') but in the aorist aspect it means "I see", "I understand", "It is now clear to me". From the many proofs of the Pythagorean Theorem to the Proofs without Words in the Mathematical Intelligencer: once a proof is given, there is insight, understanding. Theorems and proofs take us beyond real space or linear time, into the mathematical realm. This mathematical realm is closely related, in my opinion, to the notion of time, held by ancient Indo-European (and other) people.

The aorist and the perfect are widely used in Euclid. Recent research [18] suggests that Euclid uses the aorist for key verbs to denote the finalization of steps within the proof of a proposition. The aorist emphasizes the performed action and marks a new step in a proof. The perfect, on the other hand, stresses the attained result of an action and is therefore used to refer to a finished proof or a completed construction of figures. The perfect is also used to refer to earlier, already proven propositions [18]. Certainly, for Ancient Greek readers this improved the readability and intelligibility of The Elements.

Hence, interpreting the aorist as "perfect", The Elements cannot be appreciated by more modern readers. This is no surprise. In the transition from the oral to written traditions, the aorist has disappeared in almost all languages of Indo-European descent, including in Koine Greek, spoken by ancient Greek mathematicians. Together with the ascent of mathematics in Ancient Greece, languages became simpler, perhaps with purer definitions. Unfortunately, the aorist became synonym to the perfect and the aorist aspect, rooted in the notion of time and what was and what will always be, was lost.

6 Mathematics, Mushrooms and Mycelium

My paper started in well-defined space and time (visit to Göttingen, November 26-28, 2018) and wandered off into the aspect realm where space and time in itself, although unified by Minkowski at Göttingen in 1908, ceased to exist as it were.

What is always key to human culture, which includes mathematics, are deep questions and challenges, which are addressed and answered differently, in different eras and geographical locations. One recurring theme is the erasure of the past, or of other ways of doing mathematics. This is often bound up with narrowing our brain, and simplifying our language. That is how the aorist lost its original meaning, and simplifications present a continuous threat to mathematics. The geometer Radu Miron wrote: "*If Mathematics would be torn from its foundations, it would become a series of formulae, recipes and tautologies that could not be applied any longer to the objective reality, but only to some rigid, mortified scheme of this reality*" [19].

Mathematics is a lively activity, a human endeavor, and real progress is slow, when measured in solutions of problems and in theorems. In Arnold's view: progress is visible as the mushrooms, the fruiting bodies of the fungi with their spores [20]. Real and living mathematical activity however is at least equally important. For Arnold this was the mycelium of fungi, living underground. From a botanical perspective: in forests the mycelium is a crucial part of the ecosystem, providing communication channels and even distribution of nutrients for trees and plants. The underground mathematical mycelium activity generates challenges and problems, ideas, mistakes, conjectures, discoveries, as well as failure and learning from failure (Fig. 7). It is by no means simply linear. This is the legacy of Ancient Greek mathematics to this very day. To understand this legacy, we need not only to understand mathematics, but also language and human culture.

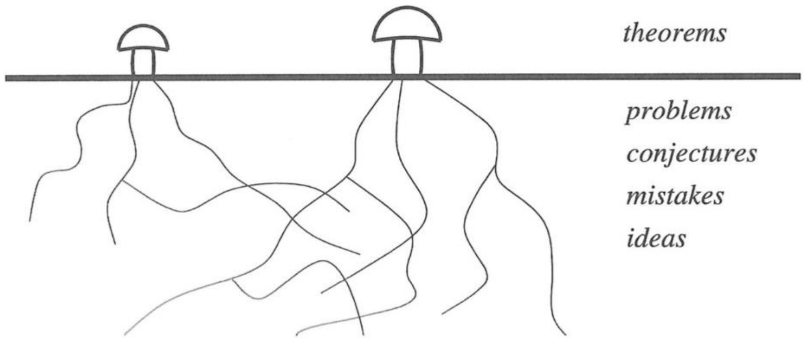


Fig. 7. Arnold's Mathematical Mushroom

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Weberian Surfaces with Foci and Directorial Planes

Maja Petrović

Abstract
The subject of this paper are algebraic surfaces named Weberian surfaces which represent loci of points with a constant sum of distances to foci and directorial planes. Algebraic equations are given for two types of these surfaces, as well as their graphical representations.

Keywords: foci, directorial planes, locus of points, algebraic 3D surfaces

Abstrakt
Predmetom tohto článku sú algebrické plochy nazývané Weberovské plochy, ktoré sú množinou všetkých bodov s konštantným súčtom vzdialeností od ohnísk a riadiacich rovín.
Pre dva typy týchto plôch sú uvedené algebrické rovnice, ako aj ich grafické znázornenie

Kľúčové slová: ohniská, riadiace roviny, množina bodov, algebrické plochy v 3D

1 Introduction

Owing to modern tools and the rapid development of technology in the XXI century, the number of newly generated algebraic surfaces with complex geometries is rising, [1], [3], [5], and their applications in various areas of engineering are expanding, [2], [4], [6]. Precisely for this reason, the subject of this paper are algebraic surfaces with predetermined 3D elements (foci and directorial planes), in regard to which they have been generated. These surfaces are called Weberian surfaces.

Weberian surfaces are defined as loci of points in space with a constant sum of scaled distances to foci, directorial lines and directorial planes, [4]. In this paper, only Weberian surfaces (WS) which represent loci of points with a constant sum of scaled distances to foci and scaled distances to directorial planes are considered. That is, Weberian surfaces with m foci and k directorial planes are analysed, which are in turn given by the following equation:

$$\sum_{i=1}^{m} \alpha_i R_i + \sum_{j=1}^{k} \beta_j d_j = S, \; S \geq 0, \qquad (1)$$

where R_i is the distance of a point of the Weberian surface from the i-th focus F_i $(i = 1 \ldots m)$ and d_j is the distance of a point of the Weberian surface from the j-th directorial plane D_j $(j = 1 \ldots k)$, while the coefficients α_i, β_j and S are real numbers.

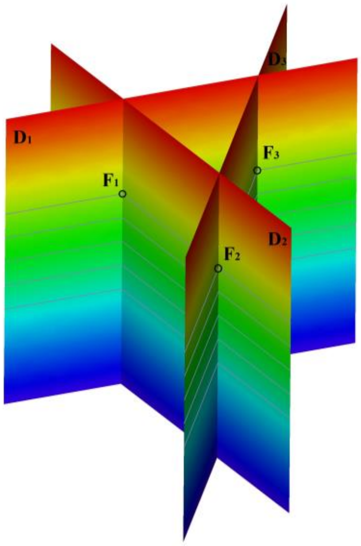
2 Determination of the basic elements of Weberian surfaces

2.1 3D elements (foci and directorial planes) for surface generation

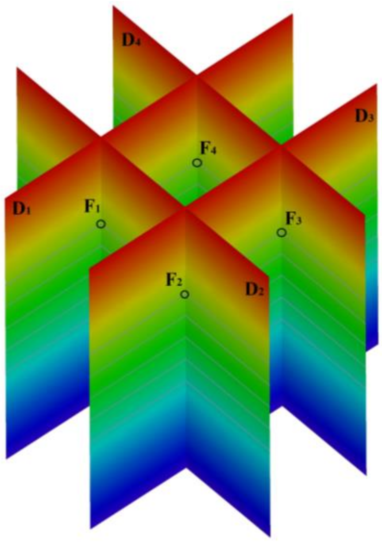
In this section of the paper, some restrictions are introduced into the equation (1). In other words, the geometric structure of the surface is defined i.e., the numbers and spatial arrangements of 3D elements (foci and directorial planes) used for generating the WS are set.

The first restriction is related to the number of foci m and the number of directorial planes k, insofar as their counts are equalized i.e., $\boldsymbol{m = k}$.

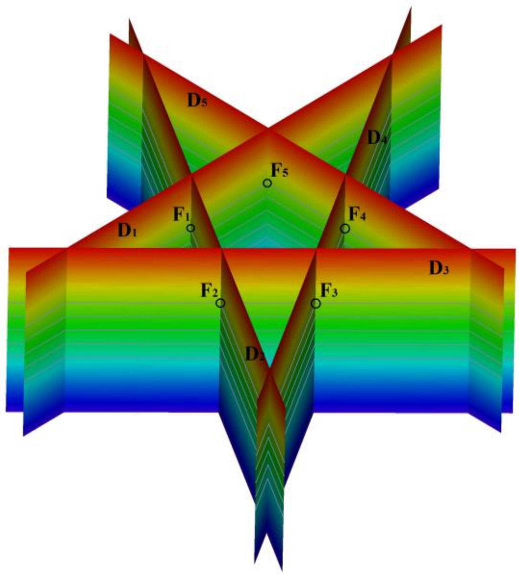
The second restriction is related to the spatial arrangements of foci and directorial planes. Let the foci F_i coincide with the vertices of an equilateral triangle, a square or a regular pentagon; that is, it holds that: $\mathbf{3} \leq \boldsymbol{m = k} \ \leq \mathbf{5}$; and let the directorial planes D_j used for generating the WS coincide with the sides of the upright prism the base of which is given by the previously defined foci, see Fig. 1.



a) $m = k = 3$;



b) $m = k = 4$;



c) $m = k = 5$

Fig. 1. The spatial arrangements of foci F_i and directorial planes D_j $(i = 1 \dots m, j = 1 \dots k)$

3D elements with such spatial arrangements (Fig. 1) are the foundational starting point for forming new algebraic surfaces.

2.2 The parameter S and the Weberian coefficients α_i, β_j for surface generation

The following restrictions are related to the numeric value of the parameter S as well as to the numeric values of the Weberian coefficients. The constant sum i.e., the parameter S is a real number necessary for the generation of WS, as are the scaled distances i.e., the Weberian coefficients α_i, β_j $(i = 1 \dots m, j = 1 \dots k)$ which are also real numbers. Based on these parameters, two groups of WS are introduced, for which it respectively holds that: *I)* $S = 0$ and *II)* $S > 0$. The definitions of these WS are as follows:

I) The WS represents a locus of points with its sum of distances to foci equal to two and a half times its sum of distances to directorial planes. This means that the parameter S for this surface is equal to zero in Eq. (1). The foci coefficients of the WS are mutually equal and it holds that $\alpha_i = 1$, while the directorial plane coefficients have the numeric value $\beta_j = -2.5$. This algebraic surface given by the equation (2) is called the *Weberian Surface type I:*

$$\sum_{i=1}^{m} R_i = 2.5 \sum_{j=1}^{k} d_j \,; \tag{2}$$

II) A WS represented as a locus of points with a constant sum ($S > 0$) of unit distances to foci ($\alpha_i = 1$) and negative distances to directorial planes ($\beta_j = -1$) has the name *Weberian Surface type II* and its equation is as follows:

$$\sum_{i=1}^{m} R_i - \sum_{j=1}^{k} d_j = S. \tag{3}$$

3 Generation of Weberian surfaces

The geometric structure of surfaces defined using equations (2) and (3) is the same, but with slight variations of initial parameters (S or Weberian coefficients), the form of the generated algebraic surfaces varies significantly. Hence, in this section of the paper, all the necessary numeric values for the aforementioned equations are given, followed by graphical displays of the newly generated surfaces.

3.1 WS with three foci and three directorial planes

The first WS to be defined has foci which coincide with the vertices of an equilateral triangle: $F_1(-1,0,0)$, $F_2(1,0,0)$, $F_3(0,\sqrt{3},0)$, while the directorial planes of this surface coincide with the sides of the upright prism with the base $F_1F_2F_3$, see Fig. 1a.

The functions of the distances of a point of the WS to foci F_i $(i = 1, \dots, 3)$ are as follows:

$$R_1 = \sqrt{(x+1)^2 + y^2 + z^2};\ R_2 = \sqrt{(x-1)^2 + y^2 + z^2};\ R_3 = \sqrt{x^2 + \left(y - \sqrt{3}\right)^2 + z^2}.$$

The functions of the distances of a point of the WS to directorial planes D_j $(j = 1, \dots, 3)$ are:

$$d_1 = \left|\frac{-y + \sqrt{3}x + \sqrt{3}}{2}\right|;\ d_2 = |y|;\ d_3 = \left|\frac{y + \sqrt{3}x - \sqrt{3}}{2}\right|.$$

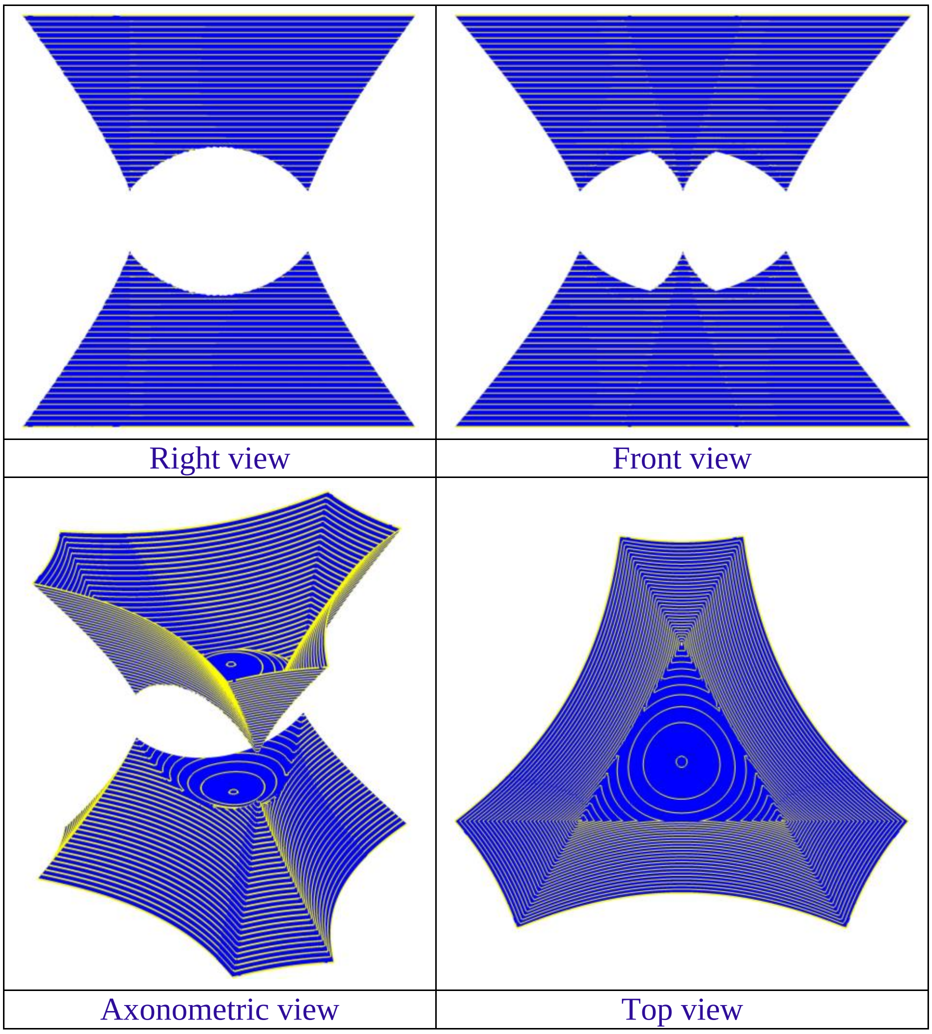


Fig. 2. WS with three foci and three directorial planes *(type I)*

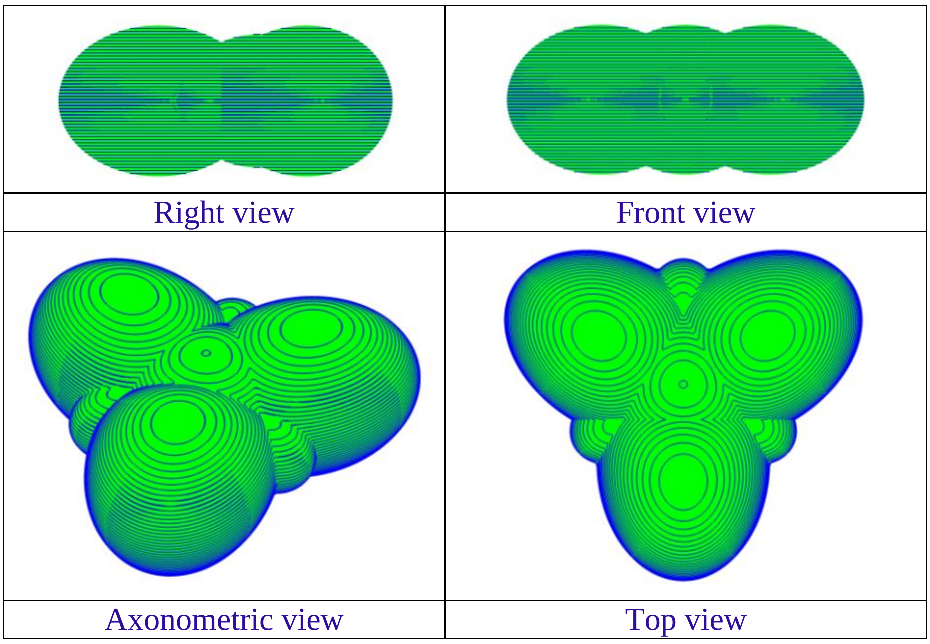


Fig. 3. WS with three foci and three directorial planes *(type II)*

Fig. 2 shows a newly generated surface given by $R_1 + R_2 + R_3 = 2.5(d_1 + d_2 + d_3)$ in all three orthogonal projections, as well as its axonometric view; while Fig. 3 shows a WS type II, the equation for which is $R_1 + R_2 + R_3 - (d_1 + d_2 + d_3) = 3$.

3.2 WS with four foci and four directorial planes

By increasing the number of 3D elements of the Weberian surface to four, and positioning the foci in such a way that they coincide with the vertices of a square: $F_1(-1,0,0)$, $F_2(1,0,0)$, $F_3(1,2,0)$, $F_4(-1,2,0)$ and the directorial planes coincide with the sides of the upright prism with the base $F_1F_2F_3F_4$, see Fig. 1b, the following functions of distances of a point of the WS to foci F_i $(i = 1, \ldots, 4)$ are obtained:

$$R_1 = \sqrt{(x+1)^2 + y^2 + z^2};\ R_2 = \sqrt{(x-1)^2 + y^2 + z^2};$$

$$R_3 = \sqrt{(x-1)^2 + (y-2)^2 + z^2};\ R_4 = \sqrt{(x+1)^2 + (y-2)^2 + z^2}.$$

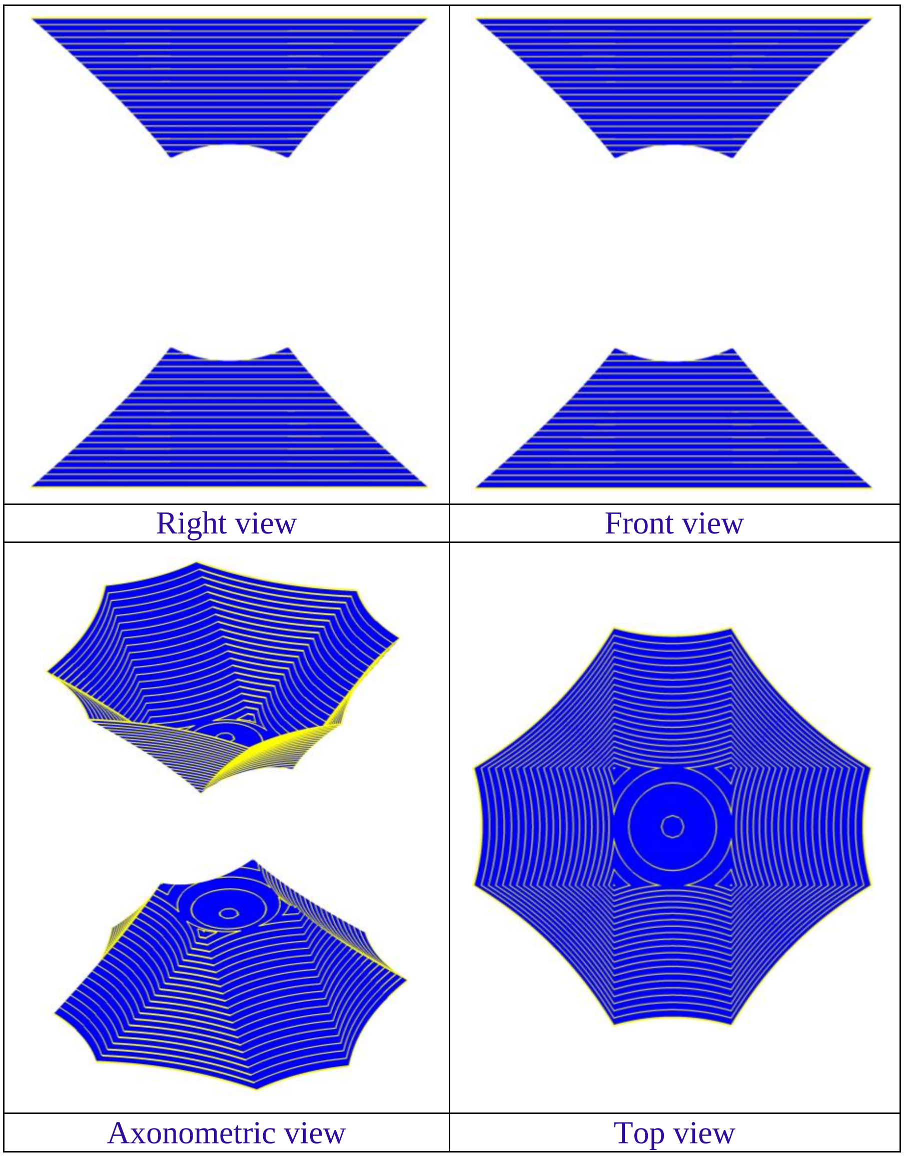


Fig. 4. WS with four foci and four directorial planes *(type I)*

The functions of distances of a point of the WS to directorial planes D_j $(j = 1, \dots, 4)$ are:

$$d_1 = |x + 1|;\; d_2 = |y|;\; d_3 = |x - 1|;\; d_4 = |y - 2|\,.$$

Fig. 4 shows the newly generated WS type I given by $R_1 + R_2 + R_3 + R_4 = 2.5(d_1 + d_2 + {} + d_3 + d_4)$ in all three orthogonal projections, as well as its axonometric view.

A graphical display (top, front, right and axonometric views) of the Weberian surface type II given by the equation $R_1 + R_2 + R_3 + R_4 - (d_1 + d_2 + d_3 + d_4) = 4$ is shown in Fig. 5.

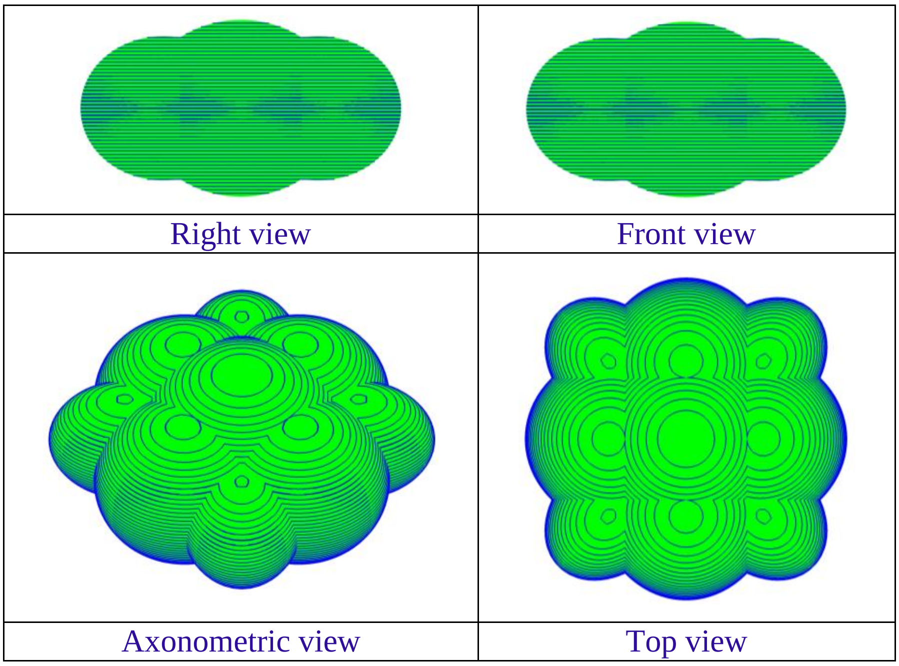


Fig. 5. WS with four foci and four directorial planes *(type II)*

The work [5] considers Weberian surfaces with four foci and four directorial planes with the same spatial arrangement of 3D elements as given in Fig. 1b) and the same Eq. (1), but with different values for the Weberian coefficients and the parameter S compared to the ones given in this section of this paper.

3.3 WS with five foci and five directorial planes

In this subsection of the paper, a third group of WS has been generated which has foci coinciding with the vertices of a regular pentagon: $F_1(-1,0,0)$, $F_2(1,0,0)$, $F_3(\phi, q, 0)$, $F_4(0, h, 0)$, $F_5(-\phi, q, 0)$, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$; $q = \sqrt{2+\phi} \approx 1.902$; $h = \sqrt{5 + 2\sqrt{5}} \approx 3.077$, while the directorial planes of this surface coincide with the sides of the upright prism with the base $F_1F_2F_3F_4F_5$, see Fig. 1c. The functions of distances of a point of the WS to foci F_i $(i = 1, \dots, 5)$ are as follows:

$$R_1 = \sqrt{(x+1)^2 + y^2 + z^2};\; R_2 = \sqrt{(x-1)^2 + y^2 + z^2};$$

$$R_3 = \sqrt{(x-\phi)^2 + (y-q)^2 + z^2};\; R_4 = \sqrt{x^2 + (y-h)^2 + z^2};$$

$$R_5 = \sqrt{(x+\phi)^2 + (y-q)^2 + z^2}.$$

The functions of distances of a point of the WS to directorial planes D_j $(j = 1, \dots, 5)$ are:

$$d_1 = \left|\frac{(\phi-1)y\ + qx + q}{2}\right|;\ d_2 = |y|;\ d_3 = \left|\frac{-(\phi-1)y\ + qx - q}{2}\right|;$$

$$d_4 = \left|\frac{\phi y\ + (h-q)x - (h+q)}{2}\right|;\ d_5 = \left|\frac{-\phi y\ + (h-q)x + h + q}{2}\right|.$$

A graphical display (top, front, right and axonometric views) of the Weberian surface type I defined by the equation $R_1 + R_2 + R_3 + R_4 + R_5 = 2.5(d_1 + d_2 + d_3 + d_4 + d_5)$ is given in Fig. 6.

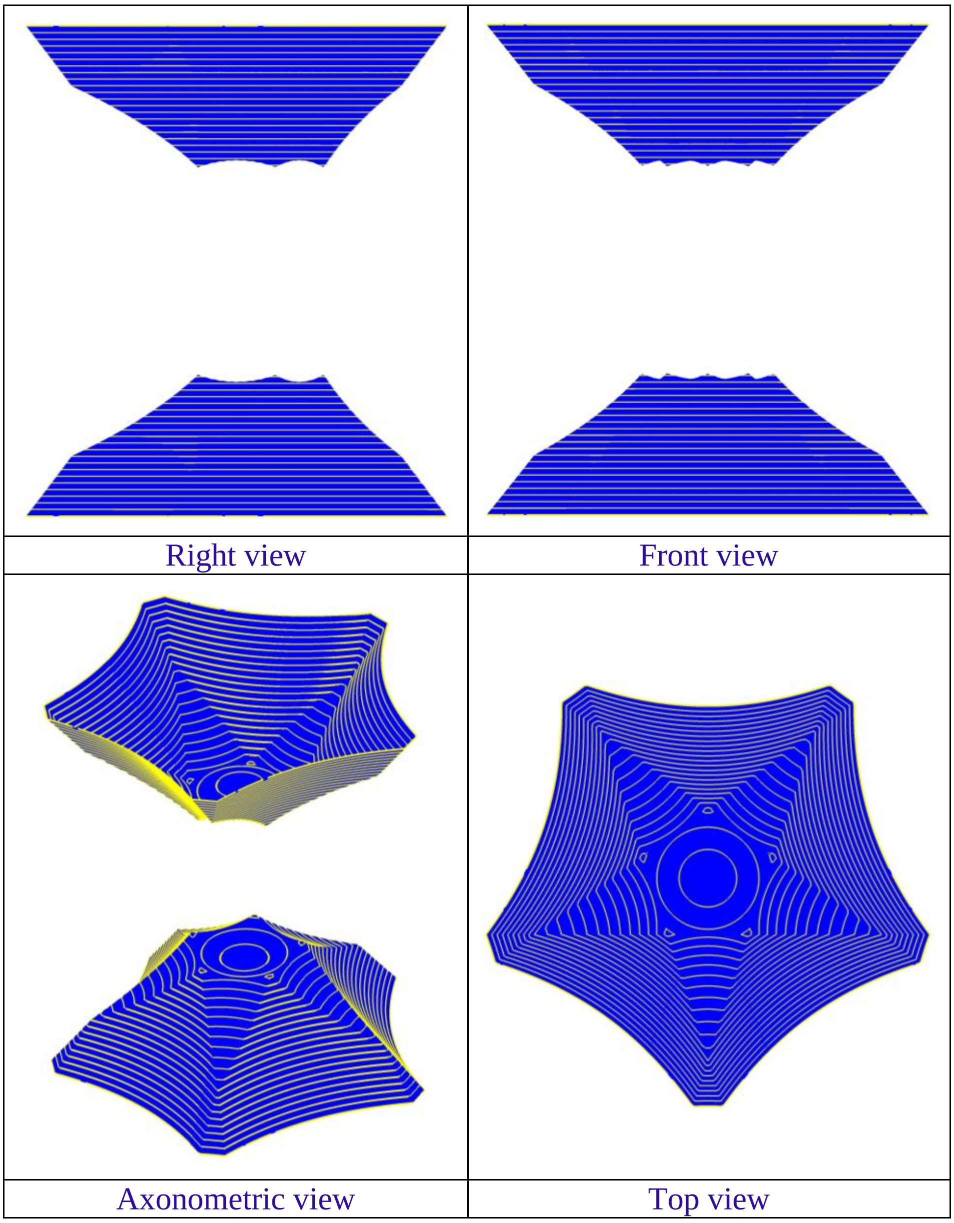


Fig. 6. WS with five foci and five directorial planes *(type I)*

The WS given by the equation $R_1 + R_2 + R_3 + R_4 + R_5 - (d_1 + d_2 + d_3 + d_4 + d_5) = 5$ is shown in Fig. 7.

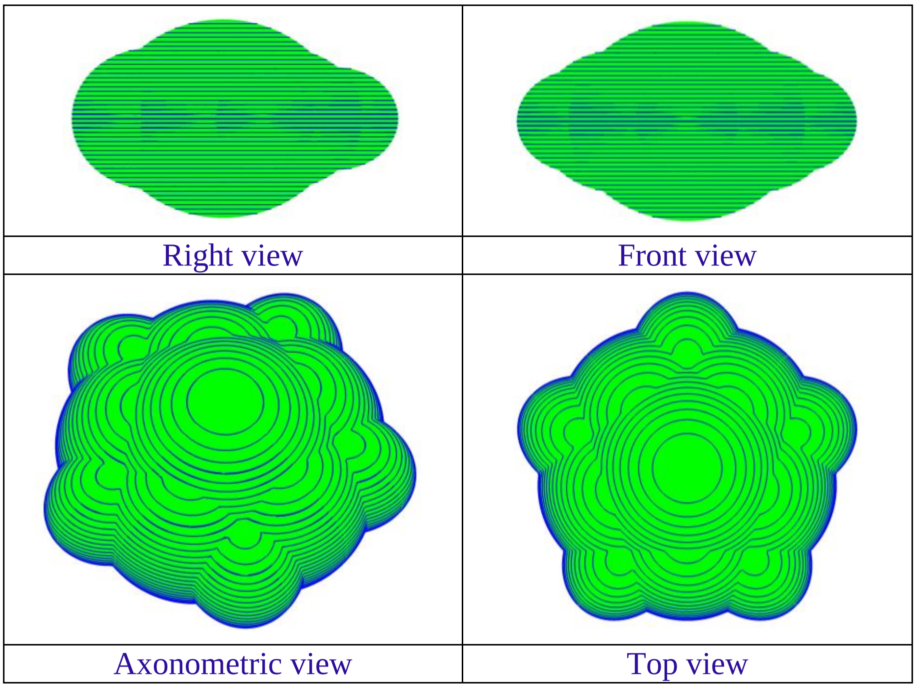


Fig. 7. WS with five foci and five directorial planes *(type II)*

The modelling of the surfaces shown in this paper (Fig. 2 - Fig. 7) was done using the free software package VisuMath 3.0. “VisuMath 3.0 is a mathematics visualization tool. It can be used to visualize curves in a plane and surfaces in the three dimensional Euclidean space“, as stated by the author of this software package Ignace Van de Woestyne, [7].

4 Conclusion and future research

Based on the previous section, the following conclusions can be drawn: WS type I surfaces are open surfaces with an “umbrella” shape and could belong to the group of hyperbolic surfaces, while type II surfaces are closed surfaces with a “pillow” shape. Furthermore, because of the spatial arrangement of their foci and directorial planes (Fig.1), these surfaces have the same symmetry as their initial 3D elements. Moreover, types of surfaces defined thusly with Eq. (2) and Eq. (3) have great adaptability and flexibility with respect to variations in the initial 3D elements (changing the spatial arrangement of foci which can coincide with the vertices of irregular polygons or changing the spatial arrangement of directorial planes in such a way, so they coincide with the sides of an oblique prism).

Modern technology and materials used in design and engineering have contributed to the possibility of realizing different, complex, mathematically-defined 3D structures. Therefore, the newly generated surfaces (Fig. 2 - Fig. 7), in part or whole, could find wide application as standalone architecturally constructed objects, and so our further research could be tied to their static structural analysis, adaptability for pneumatic purposes (for type II surfaces), ...

Acknowledgement

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[7] VisuMath 3.0 by Ignace Van de Woestyne (free software): http://www.visumath.be/

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Abstracts

J. Brajerčík: Problems of global variational geometry

We introduce the global variational geometry as a modern mathematical discipline integrating the knowledge of many areas of mathematics. It generalizes classical calculus of variations, and its subject is a geometric structure consisting of a smooth manifold endowed with a differential form. In this contribution, basic concepts of the global variational geometry are characterized. We also discuss some problems solved by methods of the global variational geometry.

J. Donnelly: The Equivalence of Side-Angle-Side and Side-Side-Angle in the Absolute Plane

In general, Side-Side-Angle is not a criterion for congruence of triangles. However, one can use Side-Angle-Side to prove several statements in triangle geometry which involve Side-Side-Angle. We prove that the converse holds as well. Namely, we show that if we assume the statements which involve Side-Side-Angle, then we can prove Side-Angle-Side as a consequence of these assumptions. In these proofs, no assumptions about Euclidean or hyperbolic parallel properties are made.

J. Gielis: A Mathematical Taorist in Göttingen

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Mathematics itself will never be servant to twitter.
Algebra, calculus & geometry, A Holy Trinity
Mathematics' own Divinity
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To establish it is rooted in language, in the Aorist

M. Petrović: Weberian Surfaces with Foci and Directorial Planes

The subject of this paper are algebraic surfaces named Weberian surfaces which represent loci of points with a constant sum of distances to foci and directorial planes. Algebraic equations are given for two types of these surfaces, as well as their graphical representations.

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